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Relations in the Tautological Ring of the Universal Curve

Master Thesis

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Abstract

After some background theory we provide a brief summary of what is known about the tautological ring of the moduli space of curves. We then formulate a few conjectures about the structure of the tautological ring of the universal curve. These conjectures are analogous to the so-called “Faber conjectures”. We verify these conjectures for genus $2 \leq g \leq 9$. We also study some matrices associated to the conjectures and find a relationship between these matrices and the corresponding matrices on the moduli space of curves.

Contents

Chapter 1
Introduction

In the article *A Conjectural Description of the Moduli Space of Curves*, [10], Faber formulates a number of conjectures concerning the tautological ring of the moduli space of curves. The aim of this project has been to formulate, and for low genera prove, similar conjectures for the universal curve over the moduli space of curves. However, before reaching this point we need quite a bit of background.

Among the possible starting points I have chosen to begin with manifolds. This might seem a bit unnatural given that this is a project in algebraic geometry. However, this approach has a few advantages. Firstly, it allows me to define the genus of a curve topologically which is often regarded as simpler than defining the geometric genus and, more so, the arithmetic genus. Secondly, the definition of a manifold requires fewer definitions than the definition of a scheme even though these objects are in many ways analogous. It might therefore be beneficial to have seen manifolds before being introduced to schemes. Thirdly, the triple equivalence between smooth projective complex algebraic curves, complex algebraic function fields and compact Riemann surfaces at least motivates a discussion of this topic at some point in this context.

We continue by discussing affine and projective varieties. These are the simplest algebraic counterparts of manifolds. In the following section we introduce vector bundles, sheaves and schemes, which are more complicated algebraic counterparts of manifolds. The discussion largely follows the one found in [21].

In Section 2.4 we introduce the Chow ring. The tautological ring will later be defined as a subring of this ring. Finally, in Section 2.5 we introduce the moduli space of curves, $\mathcal{M}_g$.

In Chapter 3 we first introduce a few objects related to the moduli space of curves, most importantly the universal curve $\mathcal{C}_g$ and the tautological rings $R(\mathcal{M}_g)$ and $R(\mathcal{C}_g)$. We then briefly discuss what is known about $R(\mathcal{M}_g)$. Finally, we formulate and discuss the so-called Faber conjectures.

Chapter 4 is about what has been done in this project. We first state analogues of the Faber-conjectures. Sections 4.1-4.3 concerns certain matrices related to the conjectures. In Section 4.4 we discuss how to generate relations in $R(\mathcal{C}_g)$. Using the relations and the matrices, the conjectures are proven for low genera. Finally, in Section 4.5 we provide some concluding remarks.

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Chapter 2
Background Theory

2.1 Manifolds and Riemann Surfaces

The concept of manifolds generalizes the concepts of curves and surfaces. A continuous curve could be defined as a topological space which locally looks like the real line. Similarly, a surface could be defined as a topological space whch locally looks like Euclidean 2-space. Therefore it seems intuitive to define a manifold as a topological space which locally looks like Euclidean n-space, for some n. However, we actually require more of a curve than just looking like $\mathbb{R}^n$ around every point.

For instance, consider the disjoint $\mathbb{R}^1 \sqcup \mathbb{R}^1$ with the ordinary topology. Define an equivalence relation $\sim$ on this space by saying $x \sim y$ if $x = y \neq 0$ as elements in $\mathbb{R}$. The space $\mathbb{R}^1 \sqcup \mathbb{R}^1 / \sim$ is a space such that every point has a neighbourhood homeomorphic to the real line, but it has "two origins". We can find an open set U containing one of the origins but not the other, but if we try to find an open set V which contains the other origin and is disjoint from U we invariably fail. Hence, $\mathbb{R}^1 \sqcup \mathbb{R}^1 / \sim$ is not Hausdorff (but it is T_1). This is not what we would call a "curve", so this might justify the following definition.

Definition 2.1. A topological manifold, M, is a paracompact Hausdorff topological space such that every point of M has a neighbourhood homeomorphic to some Euclidean space, $\mathbb{R}^n$.

Recall that a space is paracompact if every open cover admits a locally finite refinement, i.e. a refinement such that every point has a neighbourhood which only intersects a finite number of sets of the refinement. Note that in the above definition we do not require n to be constant. However, it turns out that n will be constant on the connected components of M. We shall only consider connected manifolds.

We shall always require more of manifolds than simply being topological manifolds. Therefore we make the following definition.

Definition 2.2. A chart on a topological manifold M consists of an open set $U \subset M$ and a homeomorphism $\phi : U \to V \subset \mathbb{R}^n$, where V is an open subset of $\mathbb{R}^n$. A smooth atlas on M is a collection $\mathscr{A} = \{(U_i, \phi_i)\}_{i \in I}$ of charts such that $M = \bigcup_{i \in I} U_i$, $\phi_i(U_i \cap U_j)$ is open for all $i, j \in I$ and such that each transition function

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

is smooth for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$.

We say that two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ on M are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is a smooth atlas on M. An equivalence class of smooth atlases on M is called a smooth structure on M.

Definition 2.3. A smooth manifold $(M, \mathscr{A})$ is a topological manifold M with a smooth structure $\mathscr{A}$.

By complete analogy to the above we define complex manifolds.

Definition 2.4. A complex chart on a topological manifold M is a homeomorphism $\phi : U \to V$ from an open set $U \subset M$ to an open set $V \subset \mathbb{C}^n$. A holomorphic atlas on M is a collection of complex charts $\mathscr{A} = (U_i, \phi_i)_{i \in I}$ on M such that the U_i cover M, the sets $\phi_i(U_i \cap U_j)$ are open for each $i, j \in I$ and such that the transition functions

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

are holomorphic for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. Two holomorphic atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are equivalent if $\mathscr{A}_1 \cup \mathscr{A}_2$ is a holomorphic atlas. An equivalence class of holomorphic atlases is called a complex structure. A complex manifold is a topological manifold M with a complex structure $\mathscr{A}$.

We are now ready to define Riemann surfaces.

Definition 2.5. A Riemann surface is a connected complex manifold of complex dimension one.

Since a Riemann surface has complex dimension one, it has real dimension two. This explains why we call them surfaces.

We now introduce the important class of holomorphic functions on a Riemann surface.

Definition 2.6. Let X be a Riemann surface and let V be an open subset of X. A function $f : V \to \mathbb{C}$ is called holomorphic if the functions

$$f \circ \phi_i^{-1} : \phi_i(U_i \cap V) \to \mathbb{C}$$

are all holomorphic.

Definition 2.7. The set of all holomorphic functions on $V \subset X$ is denoted by $\mathcal{O}(V)$. The set $\mathcal{O}(V)$ is a ring called the ring of holomorphic functions on V.

Another important class is the class of meromorphic functions.

Definition 2.8. Let X be a Riemann surface and let V be an open subset of X. A function $f : V \to \mathbb{C}$ is called meromorphic if there is a set of isolated points A in V such that the restriction $f : V \setminus A \to \mathbb{C}$ is a holomorphic map and such that $\lim_{x\to p} |f(x)| = \infty$ for all $p \in A$.

Definition 2.9. The set of all meromorphic functions on $V \subset X$ is denoted by $K(V)$. $K(V)$ is a field called the field of meromorphic functions on V.

Yet another important concept is that of a holomorphic map between two Riemann surfaces.

Definition 2.10. Let X and Y be Riemann surfaces with atlases (U_i, ϕ_i), (V_j, ψ_j), respectively. A map $f : X \to Y$ is holomorphic if the maps

$$\psi_j \circ f \circ \phi_i^{-1} : \mathbb{C} \to \mathbb{C},$$

are holomorphic wherever they are defined.

Definition 2.11. Let X and Y be Riemann surfaces. $f : X \to Y$ is called an analytic isomorpism (or a holomorphic isomorphism or simply an isomorphism) if f is bijective and holomomorphic with a holomorphic inverse. If there is an isomorphism $X \to Y$ we say that X and Y are isomorphic.

Let X be a Riemann surface, let p be a point on X an let (U, ϕ) be a chart around p. We may compose ϕ with a translation T so that $T \circ \phi(p) = 0$. This will also be a homeomorphism compatible with the complex structure on X. Further, we may restrict $T \circ \phi$ to an open subset U' of U so that $T \circ \phi(U')$ is an open disc with center 0. We may then rescale this disc to obtain a disc with radius 1. These are all operations compatible with the complex structure on X. Hence, we may as well assume that a chart (U, ϕ) around p maps p to 0 and that $\phi(U)$ is the open unit disc. This will be convenient in the following discussion of holomorphic and meromorphic functions.

Let f be a meromorphic function. The composition $f \circ \phi^{-1}$ can be written as

$$f \circ \phi^{-1}(z) = \sum_{i=m}^{\infty} c_i z^i,$$

for some constants $c_i \in \mathbb{C}$ and $c_m \neq 0$. The number m is called the order of f at p. If m is nonnegative, then f is holomorphic at p and if m is positive then f has a zero of multiplicity m at p. If m is negative we call p a pole of multiplicity $-m$ of f. If X is a compact Riemann surface and f is a nonzero function, then f has only finitely many zeros and poles. Further, if we denote the order of a point $p \in X$ by m_p then

$$\sum_{p_p} -m_{p_p} = \sum_{p_z} m_{p_z},$$

where the p_p's are the poles of f and the p_z's are the zeros of f. In other words, counted with multiplicities, f has equally many zeros as poles.

Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. Let x be a point of X. We may choose charts (U, ϕ) and (V, ψ) around x and $f(x)$ such that

$$\psi \circ f \circ \phi^{-1} : \mathbb{C} \to \mathbb{C},$$

is given by

$$\psi \circ f \circ \phi^{-1}(z) = z^k,$$

for some positive integer k. This is clearly an open map and ϕ and ψ are homeomorphisms and therefore open. It thus follows that f is an open map. Hence, any holomorphic map between Riemann surfaces is an open map. This observation has some important consequences.

Theorem 2.1. *If X is compact and $f : X \to Y$ is a nonconstant holomorphic map between Riemann surfaces, then f is surjective and Y is compact.*

Proof. Since X is compact, $f(X) \subset Y$ is compact, and therefore closed since Y is Hausdorff. But on the other hand, X is open so $f(X)$ is open. Hence, $f(X)$ is both open and closed. Since Y is connected, this impies that $f(X) = Y$. □

Corollary 2.1. *If X is a compact Riemann surface, then $\mathcal{O}(X) = \mathbb{C}$.*

Proof. Suppose that there were a nonconstant holomorhic map $f : X \to \mathbb{C}$. By Theorem 2.1, we have that $\mathbb{C}$ is compact. This is not the case so if f is holomorphic, then f is constant. □

As a last remark, let X and Y be Riemann surfaces and let $f : X \to Y$ be a holomorphic map. If g is a meromorphic function on Y, then the pullback $f^*g = g \circ f$ is a meromorphic function on X. It follows that if f is an isomorphism, then every meromorphic function on X is the pullback of a meromorphic function of Y, and converserly. Hence, if X and Y are isomorphic Riemann surfaces then $K(X) \cong K(Y)$. In fact, the converse is also true, [32].

Let p and q be points on a Riemann surface X. A continuous path from p to q is a continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = p$ and $\alpha(1) = q$. If $p = q$ we call α a loop based at p.

A homotopy between two paths α and β between p and q is a continuous map

$$H : [0, 1] \times [0, 1] \to X,$$

such that $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$. If there is a homotopy between α and β we write $\alpha \simeq \beta$ and say that α and β are homotopic paths. Being homotopic is an equivalence relation and the equivalence class of a path α is denoted.

If α is a path from p to q and β a path from q to r we may define their product

$$\alpha.\beta(t) = \begin{cases} \alpha(2t) & \text{if } t \leq 1/2, \\ \beta(2t-1) & \text{if } t > 1/2. \end{cases}$$

It is a path from p to r.

We now fix a point p and consider loops based at p. The product of two loops based at p is again a loop based at p. We want this product to give us a group structure on the loops based at p. Unfortunately, this is not the case (actually, all group axioms fail to hold). However, if we define a product of the homotopy classes of loops by $[\alpha].[\beta] = [\alpha.\beta]$ we do get a group structure on the set of homotopy classes of loops based at p. This group is called the fundamental group of X and is denoted $\pi_1(X, p)$. Since X is connected, choosing another base point gives an isomorphic fundamental group. Hence, we may just write $\pi_1(X)$.

The fundamental group of a topological space is homotopy invariant. In particular, homeomorphic Riemann surfaces have isomorphic fundamental groups. Suppose that X is a compact Riemann surface. All Riemann surfaces are orientable, so it follows from the classifiaction theorem of compact surfaces that X is homeomorphic to “a sphere with g handles attached to it”, where g is a nonnegative integer.

If g is equal to 0, then $\pi_1(X)$ is trivial. This is because every loop can be shrunk into a point. This fact can be reformulated as “if X is homeomorphic to a sphere, then X is simply connected”.

If g is 1, then $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. One may think about $\pi_1(X)$ as wrapping a string around a torus, the left coordinate representing the number of times the string has been wrapped around the “solid” part and the right how many complete laps arond the hole of the torus the string has made. It does not matter if we first wrap the (very elastic) string n times around the “solid” part and then smear it out m laps around the hole or the other way around. This fact is reflected in the fact that $\pi_1(X)$ is abelian.

If we increase g to 2 or higher, $\pi_1(X)$ becomes more complicated. Returning to the string analogy, this is because the manner in which we wrap the string around X matters more when X is more complicated (for instance if X is a “fat 8”). Therefore, $\pi_1(X)$ is no longer abelian. However, it is still fairly simple, it is the quotient of the free group on $2g$ elements, $F(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$, by a single relation:

$$\pi_1(X) \cong \frac{F(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)}{\left(\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}\right)}.$$

If we abelianize the first fundamental group of a Riemann surface X, we obtain what is called the first simplicial homology group of X, $H_1(X, \mathbb{Z}) = \pi_1(X, p)^{\text{ab}}$. $H_1(X, \mathbb{Z})$ is isomorphic to direct sum of an even number of copies

of $\mathbb{Z}$
$$H_1(X, \mathbb{Z}) \cong \bigoplus_{k=1}^{2g} \mathbb{Z}.$$
The number g is called the genus of the Riemann surface X. It is equal to the number of "holes" of X, [2]. The genus of a Riemann surface will be important later.

2.2 Varieties

The algebraic analogue of a manifold is a nonsingular variety. In this section we shall introduce affine and projective varieties.

Let k be an algebraically closed field and define
$$\mathbf{A}_k^n = \underbrace{k \times \cdots \times k}_{n \text{ times}}.$$
The set $\mathbf{A}_k^n$ is called affine n-space over k. Elements $P = (a_1, \ldots, a_n)$ of $\mathbf{A}_k^n$ are called points and a_i is called the ith coordinate of P.

Let $k[x_1, \ldots, x_n]$ denote the ring of polynomials in n variables over k. A polynomial f in $k[x_1, \ldots, x_n]$ defines a function from $\mathbf{A}_k^n$ to k by defining
$$f(P) = f(a_1, \ldots, a_n),$$
where a_i is the ith coordinate of P. The zero set of a polynomial f in $k[x_1, \ldots, x_n]$ is the set of points P in $\mathbf{A}_k^n$ such that $f(P) = 0$ and is denoted by $Z(f)$. Similarly, if S is any subset of $k[x_1, \ldots, x_n]$ we define the zero set of S, $Z(S)$, as the set of points P of $\mathbf{A}_k^n$ such that $f(P) = 0$ for all f in S. Suppose that f and g are polynomials in $k[x_1, \ldots, x_n]$ such that $f(P) = g(P) = 0$. Then
$$(f + g)(P) = f(P) + g(P) = 0,$$
and if h is any element of $k[x_1, \ldots, x_n]$, then
$$(h \cdot f)(P) = h(P) \cdot f(P) = 0.$$
Hence, the zero set of a subset S of $k[x_1, \ldots, x_n]$ is equal to the zero set of the ideal generated by S. By Hilbert's basis theorem, see for instance [14], $k[x_1, \ldots, x_n]$ is Noetherian. Hence, any ideal of $k[x_1, \ldots, x_n]$ is generated by a finite set of polynomials.

We define an affine algebraic set to be the zero set of a subset S of $\mathbf{A}_k^n$. By the above, we may also define an affine algebraic set as the zero set of an ideal of $k[x_1, \ldots, x_n]$ or as the zero set of a finite set of polynomials.

Let $\{A_i\}_{i \in I}$ be a collection of affine algebraic sets in $\mathbf{A}_k^n$. Suppose that $A_i = Z(S_i)$ where S_i is a subset of $k[x_1, \ldots, x_n]$. Then

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I} Z(S_i) = Z\left(\bigcup_{i\in I} S_i\right),$$

and

$$A_i \cup A_j = Z(S_i) \cup Z(S_j) = Z\left(S_i \cap S_j\right).$$

Hence, the intersection of any collection of affine algebraic sets is an algebraic set and the union of two algebraic sets is an algebraic set. Further, $\mathbf{A}_k^n = Z(\{0\})$ and $\emptyset = Z(\{1\})$ so both $\mathbf{A}_k^n$ and the empty set are affine algebraic sets. These results show that the affine algebraic sets define a topology on affine n-space. This topology is called the Zariski topology.

The easiest example is, of course, $\mathbf{A}_k^1$. A subset C of $\mathbf{A}_k^1$ is closed if and only if it is the zero set of some ideal in $k[x]$. The ring $k[x]$ is a principal ideal domain so C must be the zero set of a single polynomial, f. A non-zero polynomial has only a finite set of zeros so C must either be a finite set of points, the empty set or the whole of $\mathbf{A}_k^1$. Since k is algebraically closed it must be infinite. Hence, any two non-empty open sets have non-empty intersection. In particular, the Zariski topology is not Hausdorff on $\mathbf{A}_k^n$.

Definition 2.12. A topological space X is reducible if it can be written as a union of two non-empty, proper closed subsets. If X is not reducible it is irreducible.

Similarly, we say that a subset Y of a topological space X is reducible, resp. irreducible, if it is reducible, resp. irreducible, in the subspace topology. An important feature of irreducible spaces is that any non-empty open subset of an irreducible space is dense. If X is an affine algebraic set which is also irreducible, X is called an affine variety. An open subset of an affine variety is called a quasi-affine variety.

Just as we can investigate at which points a certain set of polynomials vanish, we may investigate which polynomials that vanish on a certain set of points. We then arrive at the definition of an ideal of a subset of $\mathbf{A}_k^n$. More formally, let Y be a subset of $\mathbf{A}_k^n$. The ideal of Y, denoted $I(Y)$, is then defined as the set of polynomials f in $k[x_1, \ldots, x_n]$ such that $f(P) = 0$ for all points P in Y. The following proposition gives a condition on when an affine algebraic set is an affine variety. For a proof, see for instance [21].

Proposition 2.1. *An affine algebraic set Y in $\mathbf{A}_k^n$ is an affine variety if and only if $I(Y)$ is a prime ideal.*

The operations “taking the zero set of an ideal” and “taking the ideal of a set” seem to be opposites. The following proposition explains the relation between the two operations further.

Proposition 2.2. *Let S_1 and S_2 be subsets of $k[x_1, \ldots, x_n]$ and Y_1 and Y_2 be subsets of $\mathbf{A}_k^n$.*

(i) *If S_1 is a subset of S_2, then $Z(S_2)$ is a subset of $Z(S_1)$.*

(ii) If Y_1 is a subset of Y_2, then $I(Y_2)$ is a subset of $I(Y_1)$.
(iii) If $\mathfrak{a}$ is an ideal of $k[x_1, \ldots, x_n]$, then $I(Z(\mathfrak{a})) = rad(\mathfrak{a})$.
(iv) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(v) $Z(I(Y_1))$ is the closure of Y_1.

For a proof, see for example [21]. If we combine the above proposition with Hilbert's nullstellensatz we get the following result.

Proposition 2.3. *There is a one-to-one, inclusion reversing correspondence between affine algebraic sets in $\mathbf{A}_k^n$ and radical ideals in $k[x_1, \ldots, x_n]$ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.*

Let $Y \subseteq \mathbf{A}_k^n$ be an affine algebraic set. Two polynomials $f, g \in k[x_1, \ldots, x_n]$ define the same function on Y if and only if $f(P) = g(P)$ for all $P \in Y$ or, in other words, if $f(P) - g(P) = (f - g)(P) = 0$ for all $P \in Y$. Hence, f and g define the same function on Y if and only if $f - g$ lies in $I(Y)$. We therefore define the affine coordinate ring of Y, denoted $A(Y)$, as

$$A(Y) = \frac{k[x_1, \ldots, x_n]}{I(Y)}.$$

By the above discussion, an element $\bar{f}$ of $A(Y)$ unambiguously defines a function from Y to k by $\bar{f}(P) = f(P)$. We remark that since $I(Y)$ is a prime ideal if and only if Y is a variety it follows that Y is a variety if and only if $A(Y)$ is an integral domain.

A topological space is called Noetherian if it satisfies the descending chain condition on its closed subsets. Of course, we might as well define a Noetherian space as a space which satisfies the ascending chain condition on its open subsets. This point of view might make the analogy to Noetherian rings clearer but is seldom used. The following example indicates why.

Let $Y_1 \supseteq Y_2 \supseteq \cdots$ be a descending chain of closed subsets of $\mathbf{A}_k^n$. Then $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$ is ascending chain of ideals in $k[x_1, \ldots, x_n]$ by Proposition 2.3. Since $k[x_1, \ldots, x_n]$ is Noetherian, the chain stabilizes for some i, so $I(Y_i) = I(Y_{i+1}) = \cdots$. By Proposition 2.3 we then have $Y_i = Y_{i+1} = \cdots$. Hence, $\mathbf{A}_k^n$ is a Noetherian space.

Proposition 2.4. *Let Y be a non-empty, closed subset of a Noetherian space. Then there are unique irreducible closed subsets $Y_1, \ldots, Y_n$ such that*

$$Y = \bigcup_{i=1}^{n} Y_i,$$

and $Y_i \not\subseteq Y_j$ if $i \neq j$.

For a proof, we again advice the reader to [21]. To us, he important consequence of the above proposition is that any affine algebraic set can be expressed uniquely as a union of varieties.

Before we change the subject to projective varieties we shall define the topological dimension of a topological space X. The topological dimension of X is defined as

$$\dim(X) = \sup\{n : \text{ there is a chain } Z_0 \subset Z_1 \subset \cdots Z_n\}$$

where each Z_i is an irreducible closed set and each inclusion is proper. We define the dimension of the empty set to be $-\infty$. The dimension of an affine variety is its dimension as a topological space.

As promised, we shall now turn our attention to projective varieties. Before reaching this definition we shall however need some preparation. First we shall define projective n-space over our algebraically closed field k.

To this end, consider affine $(n+1)$-space over k, $\mathbf{A}_k^{n+1}$. Introduce an equivalence relation $\sim$ on $\mathbf{A}_k^{n+1} \setminus \{\mathbf{0}\}$ as follows: the points $P = (a_1, \ldots, a_{n+1})$ and $P' = (a'_1, \ldots, a'_{n+1})$ in $\mathbf{A}_k^{n+1} \setminus \{\mathbf{0}\}$ are equivalent if and only if there is some non-zero $\lambda \in k$ such that

$$(a'_1, \ldots, a'_{n+1}) = (\lambda a_1, \ldots, \lambda a_n).$$

Projective n-space over k, denoted $\mathbf{P}_k^n$, is now defined as $(\mathbf{A}^{n+1} \setminus \{\mathbf{0}\})/\sim$. If P is a point in $\mathbf{P}_k^n$, then any representative $(a_1, \ldots, a_{n+1})$ of P in $\mathbf{A}_k^{n+1} \setminus \{\mathbf{0}\}$ is called a set of homogeneous coordinates for P.

Consider the polynomial $f(x_1, x_2) = 1 - x_1^2 - x_2^2 \in k[x_1, x_2]$. Then $f(1, 0) = 0$. If we choose λ such that $\lambda^2 \neq 1$ we have

$$f(\lambda, 0) = 1 - \lambda^2 \neq 0.$$

Hence, if $f \in k[x_1, \ldots, x_{n+1}]$ we cannot expect to have $f(x_1, \ldots, x_{n+1}) = f(\lambda x_1, \ldots, \lambda x_{n+1})$. Further, we cannot even say that $f(\lambda x_1, \ldots, \lambda x_{n+1}) = 0$ if $f(x_1, \ldots, x_{n+1}) = 0$. However, if f is homogeneous of degree m we have

$$f(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^m f(x_1, \ldots, x_{n+1}).$$

Hence, $f(x_1, \ldots, x_{n+1})$ is still not equal to $f(\lambda x_1, \ldots, \lambda x_{n+1})$ in general, but at least we have that $f(x_1, \ldots, x_{n+1}) = 0$ if and only if $f(\lambda x_1, \ldots, \lambda x_{n+1}) = 0$ for all $\lambda \neq 0$. In other words, the concept of "being zero on a point $P \in \mathbf{P}_k^n$" is well defined for homogeneous polynomials in $k[x_1, \ldots, x_{n+1}]$. It is important to note, however, that f is still not well defined as a function from $\mathbf{P}_k^n$ to k.

The above discussion suggests that we should not study arbitrary polynomials on $\mathbf{P}_k^n$ but homogeneous polynomials. When we convert definitions and results from the affine to the projective environment we shall therefore always use homogeneous polynomials.

As a start we define the zero set of a homogeneous polynomial $f \in k[x_1, \ldots, x_{n+1}]$ as

$$Z(f) = \{P \in \mathbf{P}_k^n : f(P) = 0\}.$$

Similarly, we define the zero set of a subset $S \subseteq k[x_1, \ldots, x_{n+1}]$ of homogeneous polynomials as

$$Z(S) = \{P \in \mathbf{P}_k^n : f(P) = 0 \text{ for all } f \in S\}.$$

Here another difference from the affine case arises. In the affine case, the zero set of a set S of polynomials is the same as the zero set of the ideal generated by S. However, even if every polynomial in S is homogeneous there is no guarantee that every polynomial in the ideal generated by S is homogeneous. Hence, this result does not extend directly.

Denote the set of homogeneous polynomials in $k[x_1, \ldots, x_{n+1}]$ of degree i by K_i. The set K_i is an abelian group under addition. The ring $k[x_1, \ldots, x_{n+1}]$ is graded

$$k[x_1, \ldots, x_{n+1}] = \bigoplus_{i=0}^{\infty} K_i.$$

We use this occasion to introduce

$$k[x_1, \ldots, x_{n+1}]^+ = \bigoplus_{i=1}^{\infty} K_i.$$

An ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_{n+1}]$ is called homogeneous if it is generated by homogeneous polynomials. Equivalently, $\mathfrak{a}$ is homogeneous if it can be decomposed as

$$\mathfrak{a} = \bigoplus_{i=0}^{\infty} \mathfrak{a} \cap K_i.$$

For a proof of the equivalence, see [14]. Since a homogeneous ideal **a** is generated by a set S of homogeneous polynomials we may define the zero set of **a** as the zero set of S.

We may now define projective algebraic sets. A subset $Y \subseteq \mathbf{P}_k^n$ is called a projective algebraic set if $Y = Z(S)$ for some set $S \subseteq k[x_1, \ldots, x_{n+1}]$ of homogeneous polynomials. Analogously to the affine case, the projective algebraic sets defines the closed sets of a topology on $\mathbf{P}_k^n$ (and the proof is very similar). This topology is called the Zariski topology on $\mathbf{P}_k^n$. We define a projective variety as an irreducible projective algebraic set. An open subset of a projective variety is called a quasi-projective variety. The dimension of a projective variety is its dimension as a topological space.

Let Y be a subset of $\mathbf{P}_k^n$. We define the homogeneous ideal of Y as ideal generated by the set

$$\{f \in k[x_1, \ldots, x_{n+1}] : f(P) = 0 \text{ for all } P \in Y\}.$$

We define the homogeneous coordinate ring, $S(Y)$, of Y as

$$S(Y) = \frac{k[x_1, \ldots, x_{n+1}]}{I(Y)}.$$

As in the affine case, there are relations between the operations "taking the zero set of" and "taking the ideal of".

Proposition 2.5. *Let S_1 and S_2 be subsets of $k[x_1, \ldots, x_{n+1}]$ of homogeneous elements and Y_1 and Y_2 be subsets of $\mathbf{P}_k^n$.*

(i) *If S_1 is a subset of S_2, then $Z(S_2)$ is a subset of $Z(S_1)$.*
(ii) *If Y_1 is a subset of Y_2, then $I(Y_2)$ is a subset of $I(Y_1)$.*
(iii) *If $\mathfrak{a}$ is a homogeneous ideal of $k[x_1, \ldots, x_{n+1}]$ such that $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = rad(\mathfrak{a})$.*
(iv) *$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
(v) *$Z(I(Y_1))$ is the closure of Y_1.*

We also have the following result which corresponds to Proposition 2.3.

Proposition 2.6. *There is a one-to-one, inclusion reversing correspondence between algebraic sets of $\mathbf{P}_k^n$ and homogeneous radical ideals of $k[x_1, \ldots, x_{n+1}]$, not equal to $k[x_1, \ldots, x_{n+1}]^+$, given by $\mathbf{P}_k^n \ni Y \mapsto I(Y)$ and $k[x_1, \ldots, x_{n+1}] \ni \mathfrak{a} \mapsto Z(\mathfrak{a})$.*

In the above correspondence, projective varieties correspond to prime ideals. As in the affine case, any projective algebraic set can be uniquely as a finite union of varieties with none of the varieties containing another If Y is a projective algebraic set and

$$Y = \bigcup_{i=1}^{n} Y_i,$$

is such a decomposition into varieties, the Y_i are called the irreducible components of Y.

Projective n-space has an open covering by affine n-spaces given given by $U_i = \mathbf{P}_k^n \setminus Z(X_i)$. Each of the U_i:s are homeomorphic to $\mathbf{A}_k^n$ via the maps

$$\phi_i(P) = \phi_i(a_1, \ldots, a_{n+1}) = \left(\frac{a_0}{a_i}, \ldots, \widehat{\frac{a_i}{a_i}}, \ldots, \frac{a_{n+1}}{a_i} \right),$$

where $(a_1, \ldots, a_{n+1})$ are homogeneous coordinates of P. Another choice of homogeneous coordinates $(a'_1, \ldots, a'_{n+1})$ differs from the original choice by a non-zero factor λ, i.e. $a'_j = \lambda a_j$ for $j = 1, \ldots, n+1$. Hence

$$\frac{a'_j}{a'_i} = \frac{\lambda a_j}{\lambda a_i} = \frac{a_j}{a_i},$$

so ϕ_i is well defined. A proof that ϕ_i is a homeomorphism can be found in [21]. It follows that any projective variety can be covered by affine varieties. Similarly, quasi-projective varieties can be covered by quasi-affine varieties.

Now that we have defined affine and projective varieties, we shall consider functions on them. A function $f : Y \to k$ on a quasi-affine variety Y is called regular at a point P in Y if there exists an open set $U \subset Y$ containing P and polynomials g and h in $k[x_1, \ldots, x_n]$ such that h is never zero on U and $f = g/h$ on U. The function f is called regular if it is regular at every point of Y.

If we want to do something similar for a quasi-projective variety Y, we run into the usual problem about homogeneity. Therefore, the corresponding definition is a little more technical. More precisely, in addition to the above we need to assume that g and h are both homogeneous polynomials of the same degree. If the degree is n we have for any non-zero λ

$$\frac{g(\lambda a_1, \ldots, \lambda a_{n+1})}{h(\lambda a_1, \ldots, \lambda a_{n+1})} = \frac{\lambda^n g(a_1, \ldots, a_{n+1})}{\lambda^n (h(a_1, \ldots, a_n)} = \frac{g(a_1, \ldots, a_n)}{h(a_1, \ldots, a_n)},$$

so g/h is really a well-defined function on U even though g and h in general are not. A regular function is continuous with respect to the Zariski topology on $k = \mathbf{A}_k^1$. If Y is a quasi-affine or quasi-projective variety, the set of regular functions on Y form a ring, $\mathcal{O}(Y)$, called the ring of regular functions on Y.

If Y is an affine variety, the ring of regular functions on Y is isomorphic to the coordinate ring of Y, i.e. $\mathcal{O}(Y) \cong A(Y)$. Moreover, there is a one-to-one correspondence between points of Y and maximal ideals in $A(Y)$ given by

$$P \mapsto \mathfrak{m}_P,$$

where $\mathfrak{m}_P$ is the ideal of regular funcitons which vanish at P.

The situation is not as nice if Y is a projective variety. In fact, $\mathcal{O}(Y) \cong k$. In other words, if f is regular everywhere on the projective variety Y, then f is a constant. Hence, any interesting function on Y has points where it is not very nicely behaved.

Another important class of functions are the morphisms. Let X and Y be quasi-affine or quasi-projective varieties (they do not need to be of the same type). A morphism from X to Y is a continuous functions φ such that the function $f \circ \varphi : \varphi^{-1}(U) \to k$ is a regular function for every open set $U \subseteq Y$ and every regular function f on U. A bijective morphism is called an isomorphism if its inverse is also a morphism.

Example 2.1. Let $f(x_1, x_2) \in k[x_1, x_2]$ and consider zero set $Y = Z(x_3 - f(x_1, x_2)) \in \mathbf{A}_k^3$. This is a variety because $x_3 - f(x_1, x_2)$ is irreducible. Consider the function $\pi : Y \to \mathbf{A}_k^2$ given by $\pi(x_1, x_2, f(x_1, x_2)) = (x_1, x_2)$. This function is clearly bijective with inverse given by

$$\pi^{-1}(x_1, x_2) = (x_1, x_2, f(x_1, x_2)).$$

Let U be an open subset of $\mathbf{A}_k^2$. The inverse image of U under π is

$$\pi^{-1}(U) = \{(x_1, x_2, f(x_1, x_2)) \in Y | (x_1, x_2) \in U\}.$$

Let g be a regular function on U. The composition $g \circ \pi : \pi^{-1}(U) \to k$ takes an element $(x_1, x_2, f(x_1, x_2))$ to $g(x_1, x_2)$. Hence, if g can be written as h_1/h_2 on $V \subseteq U$ with $h_1, h_2 \in k[x_1, x_2]$ and h_2 not vanishing on V, then $g \circ \pi$ can be written as $\widehat{h}_1/\widehat{h}_2$ on $\pi^{-1}(V)$ where $\widehat{h}_i(x_1, x_2, x_3) = h(x_1, x_2)$. Note that $\widehat{h}_2$ does not vanish on $\pi^{-1}(V)$. Hence, $g \circ \pi$ is a regular function on $\pi^{-1}(U)$, so π is a bijective morphism.

Now let U be an open subset of Y. We have

$$(\pi^{-1})^{-1}(U) = \pi(U) = \{(x_1, x_2) | (x_1, x_2, f(x_1, x_2)) \in U\}.$$

Let g be a regular function on U so that g. Then, if g can be written as h_1/h_2 on some open subset $V \subseteq U$, with $h_1, h_2 \in k[x_1, x_2, x_3]$ and h_2 nowhere vanishing on V, then $g \circ \pi^{-1}$ can be written as $h_1(x_1, x_2, f(x_1, x_2))/h_2(x_1, x_2, f(x_1, x_2))$ on $\pi(V)$ and $h_2(x_1, x_2, f(x_1, x_2))$ does not vanish on $\pi(V)$. Hence, π^{-1} is also a morphism, so Y is isomorphic to $\mathbf{A}_k^n$.

Let Y be a quasi-affine or quasi-projective variety, let P be a point of Y Consider pairs $\langle U, f\rangle$ where U is an open subset of Y containing P and f is a regular function on U. Two pairs are $\langle U, f\rangle$ and $\langle V, g\rangle$ considered to be equivalent if $f = g$ on $U \cap V$. Such an equivalence class is called a germ of functions at P. We shall denote the germ containing $\langle U, f\rangle$ by $[\langle U, f\rangle]$. The set of germs of functions at P is forms a ring, denoted $\mathcal{O}_P(Y)$, where $[\langle U, f\rangle] + [\langle V, g\rangle] = [\langle U \cap V, f + g\rangle]$ and $[\langle U, f\rangle] \cdot [\langle V, g\rangle] = [\langle U \cap V, f \cdot g\rangle]$. If $[\langle U, f\rangle]$ is such that $f(P) \neq 0$, f is non-zero on some open subset V of Y containing P. Hence, $1/f$ is regular on V and $[\langle U, f\rangle] \cdot [\langle V, 1/f\rangle] = [\langle U \cap V, 1\rangle]$. Hence, if $f(P) \neq 0$ then $[\langle U, f\rangle]$ is a unit in $\mathcal{O}_p(Y)$. On the other hand, the set of germs $[\langle U, f\rangle]$ such that $f(P) = 0$ clearly defines an ideal. Hence, this ideal is a unique maximal ideal so $\mathcal{O}_P(Y)$. Therefore, $\mathcal{O}_P(Y)$ is called the local ring of Y at P.

We may also consider pairs $\langle U, f\rangle$ where U is a non-empty open subset of Y and f is a regular function on U under the same equivalence relation as above (i.e. we no longer require U to contain any specific point). The set of such functions is denoted $K(Y)$. If $[\langle U, f\rangle]$ is such that f is not identically zero, then f is non-zero on some open subset V of Y so $1/f$ is regular on V. Since both U and V are non-empty and open they are both dense so $U \cap V$ is non-empty. Since $[\langle U, f\rangle] \cdot [\langle V, 1/f\rangle] = [\langle U \cap V, 1\rangle]$ we conclude that $K(Y)$ is a field. The field $K(Y)$ is called the function field of Y and elements of $K(Y)$ are called rational functions on Y.

We conclude this section with yet another construction rather similar to the above constructions. Let X and Y be quasi-affine or quasi-projective varieties (they do not need to be of the same type). Consider pairs $\langle U, \phi\rangle$ where U is a non-empty subset of X and ϕ is a morphism from U to Y. We say that two pairs $\langle U, \phi\rangle$ and $\langle V, \psi\rangle$ are equivalent if ϕ and ψ agree on $U \cap V$. The fact that this is an equivalence relation is proved in [21]. An equivalence class of the above type is called a rational map from X to Y. A rational map $[\langle U, \phi\rangle]$ is called dominant if $\phi(U)$ is dense in Y. A rational map $[\langle U, \phi\rangle]$

from X to Y such that there is a rational map $[\langle V, \psi \rangle]$ from Y to X such that $\phi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \phi = \mathrm{id}_X$ as rational maps is called a birational map. If there is a birational map between X and Y, X and Y are called birationally equivalent.

2.3 Vector Bundles, Sheaves and Schemes

Let M be a smooth manifold. A smooth family of real vector spaces parametrized by M consists of a smooth manifold E and a smooth map

$$\pi : E \to M,$$

such that $\pi^{-1}(x)$ is a real vector space for every $x \in M$. The vector space $\pi^{-1}(x)$ is called the fibre of x. Suppose that

$$\rho : F \to M$$

is another family of real vector spaces parametrized by M. A homomorphism between (E, π) and (F, ρ) is a map

$$\phi : E \to F,$$

such that $\rho(\phi(y)) = \pi(y)$ for all $y \in E$, and such that $\phi|_{\pi^{-1}(x)} : E \to F$ is a linear map for all $x \in M$. A homomorphism ϕ is an isomorphism if there is a homomorphism $\psi : F \to E$ such that $\psi \circ \phi = \mathrm{id}_E$ and $\phi \circ \psi = \mathrm{id}_F$.

Definition 2.13. A smooth family (E, π) of real vector spaces over a smooth manifold M is locally trivial if every point $x \in M$ has a neighbourhood U such that

$$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U,$$

is isomorphic to the family pr : $U \times \mathbb{R}^k \to U$ for some positive integer k. Here pr denotes projection onto the first coordinate.

We may now define vector bundles.

Definition 2.14. A vector bundle on a smooth manifold M is a locally trivial smooth family of vector spaces

$$\pi : E \to M.$$

If U is an open set such that π is trivial when restricted to U, we have that the family $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is isomorphic to the family pr : $U \times \mathbb{R}^k \to U$ for some k. This integer k is called the rank of the vector bundle (E, π).

Example 2.2. Let M be a smooth manifold and let p be a point in M. Denote by $\mathcal{E}_p$ the algebra of germs of smooth functions at p. A derivation at p is an

$\mathbb{R}$-linear map $D_p : \mathcal{E}_p \to \mathbb{R}$ such that $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$. The tangent space of M at p, T_pM, is then defined as the vector space of all derivations at p. The tangent space of M is defined as

$$TM = \bigcup_{p \in M} \{p\} \times T_pM.$$

We have a map $\pi : TM \to M$, sending a point of TM to its first coordinate. The family (TM, π) is then a vector bundle.

Example 2.3. Consider $\mathbf{P}^n_{\mathbb{C}}$ and the set

$$E = \{([l], x) \in \mathbf{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} : x \in l\}.$$

The map

$$\begin{aligned} \pi : E &\to \mathbf{P}^n_{\mathbb{C}}, \\ ([l], x) &\mapsto [l], \end{aligned}$$

then gives (E, π) the structure of a vector bundle. The vctor bundle (E, π) is called the tautological line bundle on $\mathbf{P}^n_{\mathbb{C}}$.

Definition 2.15. Let $\pi : E \to M$ be a vector bundle and let U be an open subset of M. A section on U is a smooth function

$$s : U \to E,$$

that maps each x in U into its fibre, i.e. $\pi(s(x)) = x$.

Let s and t be two sections on U. Since $\pi^{-1}(x)$ is a vector space, $s(x)+t(x)$ is an element of $\pi^{-1}(x)$. Similarly, $c \cdot s(x)$ is an element of $\pi^{-1}(x)$ for any scalar $c \in \mathbb{R}$. It is now easy to see, although a bit tedious to write out, that the set of sections on U forms a real vector space. In fact, we may allow multiplication by any smooth function on U. The set of sections on U then becomes a module over the ring of smooth functions on U.

One may also define vector bundles over general topological spaces.

Definition 2.16. Let X be a topological space. A real vector bundle over X is a pair (E, π) where E is a topological space and π is a continuous surjection $E \to X$ such that each for each $p \in X$, $\pi^{-1}(p)$ is a real vector space, and such that there is a neighbourhood U containing p such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{R}^k$ for some nonnegative integer k.

In an analogous way one may define complex vector bundles. Vector bundles are examples of sheaves. Before defining a sheaf, we shall define a presheaf.

Definition 2.17. Let X be a topological space. Let $\mathcal{F}$ be a way of assigning to each open set U of X a set $\mathcal{F}(U)$. We require that for each pair of open sets

$U_1 \subseteq U_2$ of X there is a map $\text{res}_{U_2,U_1} : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$ (called a restriction map) such that

(*i*) $\text{res}_{U,U} = \text{id}_{\mathcal{F}(U)}$ and
(*ii*) if $U_1 \subseteq U_2 \subseteq U_3$, then $\text{res}_{U_3,U_1} = \text{res}_{U_2,U_1} \circ \text{res}_{U_3,U_2}$.

Then $\mathcal{F}$ and the system res is called a presheaf on X.

Definition 2.18. A presheaf is a sheaf if, in additon to the above, for each open U we have

(*i*) If x_1 and x_2 lie in $\mathcal{F}_U$ and $\text{res}_{U,U_i}(x_1) = \text{res}_{U,U_i}(x_2)$ for all $U_i \subseteq U$, then $x_1 = x_2$.
(*ii*) If a collection $x_i \in \mathcal{F}(U_i)$, $U_i \subseteq U$, satisfies $\text{res}_{U_i,U_i \cap U_j}(x_i) = \text{res}_{U_j,U_i \cap U_j}(x_j)$ for all i and j, then there is an x in $\mathcal{F}(U)$ such that $\text{res}_{U,U_i}(x) = x_i$.

As defined, $\mathcal{F}(U)$ is only required to be a set. However, in most cases $\mathcal{F}(U)$ has more structure. For instance, if E is a vector bundle we may take $\mathcal{F}(U)$ to be the set of all sections $s : U \to \pi^{-1}(U)$. Then each $\mathcal{F}(U)$ is a vector space and $\mathcal{F}$ is called a sheaf of vector spaces. In the same spirit, one speaks of sheaves of vector modules, sheaves of abelian groups and so on.

If $\mathcal{F}_0$ is a presheaf we can get a sheaf $\mathcal{F}$ by requiring that $\mathcal{F}$ is a sheaf such that every map from $\mathcal{F}_0$ to a sheaf factors uniquely through $\mathcal{F}$. In other words, we have a map $f : \mathcal{F}_0 \to \mathcal{F}$ such that if $\mathcal{F}'$ is another sheaf and $g : \mathcal{F}_0 \to \mathcal{F}'$ is another map, then there is a unique map $h : \mathcal{F} \to \mathcal{F}'$ such that $g = h \circ f$. The sheaf $\mathcal{F}$ is called the sheafification of $\mathcal{F}_0$.

Let X be a topological space and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of functions, one for each open set U of X, such that

$$\text{res}_{U,V} \circ \phi(V) = \phi(U) \circ \text{res}_{U,V}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves with values in the same category (i.e. such that $\mathcal{F}$ and $\mathcal{G}$ gives objects with the same type of structure) then we also require $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ to be a morphism with respect to the particular structure in question.

Let X and Y be topological spaces, let $\mathcal{F}$ be a sheaf on X and let $f : X \to Y$ be a continuous map. We can define a sheaf $f_*\mathcal{F}$ on Y by setting $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. The sheaf $f_*\mathcal{F}$ is called a direct image sheaf. If $\mathcal{G}$ is a sheaf on Y we can define a sheaf $f^{-1}\mathcal{G}$ on X as the sheafification of the presheaf which takes an open set $U \subseteq X$ to

$$\lim_{f(U) \subseteq V} \mathcal{G}(V).$$

$f^{-1}\mathcal{G}$ is called an inverse image sheaf. In particular, let $x \in X$ be a point and let $i : x \to X$ be the inclusion. The inverse image sheaf $i^{-1}\mathcal{F}$ on x is called the stalk of $\mathcal{F}$ at x and is denoted $\mathcal{F}_x$.

Recall that if A is a ring, then Spec(A) is the set of all prime ideals of A. Spec(A) can be given a topology where a subset $S \subset \text{Spec}(A)$ is closed if it is the set of prime ideals containing some ideal I. This topology is called the Zariski topology on Spec(A). For each $\mathfrak{p} \in A$ we may construct the local ring $A_{\mathfrak{p}}$. If U is an open subset of Spec(A), we define a ring

$$\mathcal{O}(U) = \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}. \tag{2.1}$$

If V is a subset of U, there is a natural restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ which is a homomorphism of rings. The assignment $U \mapsto \mathcal{O}(U)$ together with the restriction maps is in fact a sheaf of rings on Spec(A). The topological space Spec(A) together with the sheaf $\mathcal{O}$ is called the spectrum of A.

More generally, a ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ), where $f : X \to Y$ is a continuous map and ϕ is a collection of ring homomorphisms, one for each open set U in Y, satisfying $\text{res}_{f^{-1}(U), f^{-1}(V)} \circ \phi(U) = \phi(V) \circ \text{res}_{U,V}$ for all open sets $V \subseteq U \subseteq Y$. A ringed space is called a locally ringed space if for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced morphisms of stalks

$$\phi_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p},$$

map the maximal ideal of $\mathcal{O}_{Y,f(p)}$ to the maximal ideal of $\mathcal{O}_{X,x}$. Of course, an isomorphism (of ringed or locally ringed spaces) is a morphism with a two-sided inverse which is a morphism.

For instance, the spectrum of a ring is a locally ringed space.

Definition 2.19. An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$, which is isomorphic, as a locally ringed space, to the spectrum of a ring. A locally ringed space $(X, \mathcal{O}_X)$ is a scheme if every point $p \in X$ has an open neighbourhood U such that the locally ringed space $(U, \mathcal{O}_X|_U)$ is an affine scheme. A morphism of schemes is just a morphism of locally ringed spaces.

If k is a field, then Spec(k) is an affine scheme. Its underlying topological space consists of the single point (0). Perhaps more interesting is the spectrum of $k[x]$. If we assume that k is algebraically closed, then this is a scheme where the underlying topological space consists of the ideals $(x - a)$, $a \in k$, and the ideal (0). Thus, we have a one-to-one correspondence between the points $a \in \mathbf{A}_k^1$ and the closed points $x - a \in$ Spec$(k[x])$. The point $(0) \in$ Spec$(k[x])$ does not correspond to a point in $\mathbf{A}_k^1$ but rather to the whole space $\mathbf{A}_k^1$. This is reflected in the fact that the closure of the point (0) in Spec$(k[x])$ is the whole space. (0) is therefore called a generic point. Since we have an "almost perfect" correspondence between points in $\mathbf{A}_k^1$ and points of Spec$(k[x])$, the scheme Spec$(k[x])$ is called the affine line over k and also denoted $\mathbf{A}_k^1$.

Definition 2.20. Let $f : X \to Y$ be a morphism of schemes. A morphism f is called a closed immersion if $f : X \to f(X)$ is a homeomorphism, $f(X)$ is a closed subset of Y and f is an epimorphism as a morphism of sheaves.

Definition 2.21. A morphism $f : X \to Y$ is locally of finite type if Y may be covered by open affine subsets $V_i = \text{Spec}(B_i)$ such that $f^{-1}(V_i)$ may be covered by open affine subsets $U_{i,j} = \text{Spec}(A_{i.j})$, where $A_{i,j}$ is a finitely generated B_i-algebra. If each $f^{-1}(V_i)$ can be covered by a finite number of the $U_{i,j}$, f is said to be of finite type.

Definition 2.22. Let S be a scheme and let X and Y be schemes over S. The fibre product, $X \times_S Y$, of X and Y over S is a scheme with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which commute with the morphisms $X \to S$ and $Y \to S$, such that given a scheme Z with morphisms $f : Z \to X$ and $g : Z \to Y$ there is a unique morphism $\phi : Z \to X \times_S Y$ such that $f = p_1 \circ \phi$ and $g = p_2 \circ \phi$, i.e. the following diagram commutes

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr] \arrow[drrr] & & X \times_S Y \arrow[dl] \arrow[dr] & \\
& X \arrow[dr] & & Y \arrow[dl] \\
& & S &
\end{tikzcd}$$

It is not obvious that the fibre product of any two schemes exists. For a proof of the fact that it does exist, see [21]. Given that $X \times_S Y$ exists, it is not too hard to see that it is unique up to isomorphism. It is also true that the fibre product over a fixed base scheme is associative.

One application of the fibre product is the following. Let S be a scheme and let $S' \to S$ and $X \to S$ be schemes over S. Define $X' = X \times_S S'$. Then X' is a scheme over S' and we say that X' is obtained from X by making the base extension $S' \to S$.

Definition 2.23. Let $f : X \to Y$ be a morphism of schemes. Define the diagonal morphism

$$\Delta : X \to X \times_Y X,$$

as the morphism such that $p_1 \circ \Delta = p_2 \circ \Delta = \text{id}$. The morphism f is called separated if Δ is a closed immersion and X is then said to be separated over Y. If X is separated over $\text{Spec}(\mathbb{Z})$, X is simply said to be separated.

A morphism $f : X \to Y$ is said to be closed if the image of any closed subset of X is closed in Y. The morphism f is said to be universally closed if for any morphism $Y' \to Y$, the corresponding morphism $f' : X' \to Y'$, obtained by base extension, is closed.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U \subset X$ and such that if $V \subset U$, then the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A morphism of $\mathcal{O}_X$-modules is a morphism of sheaves which is also homomorphism of $\mathcal{O}_X(U)$-modules for each open set $U \subset X$. An $\mathcal{O}_X$-module, $\mathcal{F}$, is free if $\mathcal{F} = \oplus_{i \in I} \mathcal{O}_X$. The rank of $\mathcal{F}$ is the cardinality of I. The sheaf $\mathcal{F}$ is locally free if X can be covered by open sets U_j such that $\mathcal{F}|_{U_j}$ is a free $\mathcal{O}_X|_{U_j}$-module for all j. If X is connected, then the rank of $\mathcal{F}_{U_j}$ is the same for all j so we may then define the rank of a locally free sheaf as the rank of its restriction to a trivializing open set.

If $(X, \mathcal{O}_X)$ is a scheme, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called quasi-coherent if X can be covered by open sets $U_i \cong \text{Spec}(A_i)$ such that for each i, $\mathcal{F}|_{U_i}$ is an A_i-module. If, further, each $\mathcal{F}|_{U_i}$ can be chosen to be a finitely generated A_i-module, $\mathcal{F}$ is called coherent.

Before we move on we make some final definitions. A scheme X is reduced if $\mathcal{O}_X(U)$ is a reduced ring for every open set $U \subseteq X$. Recall that a ring is reduced if it contains no non-zero nilpotents. The scheme X is Noetherian if $\mathcal{O}_X(U)$ is Noetherian for all open sets $U \subseteq X$. A scheme is irreducible if its underlying topological space is irreducible. Finally, a scheme is called integral if it is both reduced and irreducible.

2.4 The Chow Ring

Let X be a integral scheme of finite type over an algebraically closed field k. The group of cycles on X, $Z(X)$, is the free abelian group generated by the closed, integral subschemes of X. The group $Z(X)$ is graded by dimension so

$$Z(X) = \bigoplus_{k=0}^{n} Z_k(X),$$

where $Z_k(X)$ is the free group generated by closed integral subschemes of dimension k and n is the dimension of X. The group $Z_k(X)$ is called the group of k-cycles on X. An element $D = \sum a_i Y_i$ in $Z(X)$ is called effective if all the a_i are non-negative. An element of $Z_{n-1}(X)$ is called a divisor. The degree of a divisor $\sum a_i Y_i$ is defined as $\sum a_i$.

Let Y be a closed, integral subscheme of dimension $k+1$ of a integral and Noetherian scheme X. Let $Y_1, \ldots, Y_m$ be the set of k-dimensional subvarieties of Y. For each Y_i, define l_i as the length of the ring $\mathcal{O}_Y(Y_i)$. Since X is Noetherian, each l_i is finite. Hence, we may associate an effective cycle

$$\langle Y \rangle = \sum_{i=1}^{m} l_i Y_i,$$

to any closed, integral subscheme Y of X.

Similarly, we may associate divisors to rational functions. Let X be a Noetherian, integral, separated scheme and let $Y \subset X$ be a closed, integral subscheme of codimension 1. Then Y has a unique generic point P (i.e. the closure of P is the whole of Y) and the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring whose quotient field is the function field of X. Let v_Y be the corresponding valuation. If f is a nonzero rational function on X, then $v_Y(f)$ is an integer. If $v_Y(f)$ is positive, f is said to have a zero of order $v_Y(f)$ along Y. Similarly, if $v_Y(f)$ is negative, f is said to have a pole of order $-v_Y(f)$ along Y. (For proofs, see [21]).

If X is a Noetherian, integral, separated scheme and f is a rational function on X, then $v_Y(f) = 0$ for all but possibly finitely many closed, integral subschemes Y of codimension 1 in X. Hence, we may define a divisor

$$(f) = \sum_{Y \subset X} v_Y(f) \cdot Y,$$

where the sum is over all closed, integral, codimension 1 subschemes of X. A divisor which can be written as (f) for some rational function f is called a principal divisor. If D and D' are divisors such that $D - D'$ is principal, then D and D' are called linearly equivalent. (For proofs, see [21]).

More generally, let L be a line bundle on X and let s be a section of L. We may choose a trivializing cover, $\{U_i\}$, of X so that on each U_i, s is given by a function f_i. We may then define a divisor associated to s as

$$(s) = \sum_{Y \subset X} v_Y(f_i) \cdot Y,$$

where the sum is over all closed, integral, codimension 1 subschemes of X and, for each Y, we choose a U_i so that $Y \subset U_i$.

Define $\mathrm{Rat}(X)$ as the group generated by cycles of the form

$$\langle W \cap \{t_1\} \times X \rangle - \langle W \cap \{t_2\} \times X \rangle,$$

where t_1 and t_2 are elements of $\mathbf{P}^1_k$ and W is a subscheme of $\mathbf{P}^1_k \times_{\mathrm{Spec}(\mathbb{Z})} X$ not contained in $\{t\} \times X$ for any $t \in \mathbf{P}^1_k$. We then define two cycles, Y_1 and Y_2, to be rationally equivalent if $Y_1 - Y_2$ lies in $\mathrm{Rat}(X)$.

Rational equivalence is an equivalence relation on $Z(X)$. We detote the equivalence class of a cycle Y by $[Y]$ and define the Chow group of X as

$$A(X) = \frac{Z(X)}{\mathrm{Rat}(X)}.$$

The Chow group is graded by dimension

$$A(X) = \bigoplus_{k=0} A_k(X),$$

where $A_k(X)$ is the group of rational equivalence classes of k-cycles.

In fact, we can make the Chow group into a ring in a very geometrical way. Let Y and Z be two closed subschemes of of a integral sheme X. We also assume that X is of finite type over an algebraically closed field k. The subschemes Y and Z are said to intersect transversally at a point $p \in X$ if X, Y and Z are all smooth at p and $T_pY + T_pZ = T_pX$, i.e. the tangent space of X is the sum of the tangent space of Y and the tangent space of Z. The suschemes Y and Z are generically transverse if Y and Z intersect transversally at each point of some open (and thus dense) subset of each component of $Y \cap Z$. One may then define a product on $A(X)$ as

$$[Y] \cdot [Z] = [Y \cap Z],$$

and then extend by linearity. This makes $A(X)$ into a commutative ring called the Chow ring. See [7] or [15] for a more complete discussion on this subject.

Intersecting something of codimension i with something of codimension j in most cases gives something in codimension $i+j$. In fact, if X has dimension n and $[Y] \in A_{n-i}(X)$ and $[Z] \in A_{n-j}(X)$, then $[Y] \cdot [Z] \in A_{n-i-j}(X)$ (this is due to Chow's "Moving Lemma", see [7]). Hence, when considering the ring structure on the rational equivalence classes, it is more convenient to consider the codimension than the dimension. If we define $A^k(X) = A_{n-k}(X)$ we have

$$A(X) = \bigoplus_{k=0}^{n} A^k(X).$$

Since $A^i(X) \cdot A^j(X) \subseteq A^{i+j}(X)$, this grading makes the Chow ring into a graded ring.

Before ending this section we shall give a comment on the so-called Chern classes. Let X be an integral scheme of finite type over an algebraically closed field k, let L be a line bundle on X and let s be a rational section of L. As stated earlier, we may define an associated divisor, (s), to s. If t is another rational section of L then s/t is a rational function and therefore $(s/t) = (s) - (t)$ is a principal divisor. Principal divisors lie in $\mathrm{Rat}(X)$ so so s and t define the same class in $A^1(X)$. We define the first Chern class of L, denoted $c_1(L)$, as the class defined by (s) for any rational section of L.

There are also other Chern classes, $c_i(L) \in A^i(X)$, and one may also define Chern classes for more general bundles. However, this requires quite a bit more work and we therefore refer the reader to [7], [15] or [21]. We remark that the total Chern class of a vector bundle, V, is defined as

$$c(V) = c_0(V) + c_1(V) + \cdots + c_r(V),$$

where r is the rank of the vector bundle.

2.5 Moduli Spaces of Curves

As a set, the moduli space of curves is the collection of all isomorphism classes of algebraic curves. However, there are several things to note. Firstly, a curve C of a given genus can only be isomorphic to another curve of the same genus. (We have only given a defintion of the so-called topological genus. There are other definitions, most importantly the so-called geometric genus. For a 1-dimensional complex manifold M, this is defined as the dimension of the space of holomorphic 1-forms on M. There is also something called the arithmetic genus, but this requires a bit more work to define). Hence, the moduli space is a disjoint union of the moduli spaces of curves of different genera. Secondly, if one thinks about algebraic curves rather concretely as the zero set of some polynomial, one might expect to obtain a continuously varying family of curves as one alters the coefficients in a contiuous manner. More topologically, we may think about this process as deformations of one curve into another. From either point of view, it seems desirable to impose some structure on the moduli space which captures this behaviour.

More precisely, we define a non-singular curve to be a one-dimensional, integral, separated scheme of finite type over an algebraically closed field k. A family of curves of genus g over a scheme B is a scheme X together with an epimorphism

$$\pi : X \to B,$$

such that $X_p := \pi^{-1}(p)$ is a curve of genus g for each p in the base scheme B. We say that the family is parametrized by B. Two families, (X, π) and (X', π'), over B are isomorphic if there is an isomorphism $\phi : X \to X'$ such that $\pi' \circ \phi = \pi$.

For technical reasons, the above definition does not work if $g < 2$. However, since we shall only consider the case $g \geq 2$, this definition is sufficient for our needs.

Since we have noted that two curves of different genera cannot be isomorphic, we consider the set of isomorphism classes of curves of some fixed genus g over an algebraically closed field k. We denote this set by $\mathcal{M}_g$. Given a family of curves of genus g, $\pi : X \to B$, we can define a function

$$\phi_{B,X} : B \to \mathcal{M}_g,$$
$$\phi_{B,X}(p) = [X_p].$$

We want to impose a scheme structure on $\mathcal{M}_g$ such that $\phi_{B,X}$ is a morphism.

Definition 2.24. Let S denote the set of isomorphy classes of curves of genus g. Let $\mathcal{M}_g$ be a scheme such that there is a bijection $\psi : S \to \mathcal{M}_g$ such that

(*i*) for every family $\pi : X \to B$, the composition $\psi \circ \phi_{B,X} : B \to \mathcal{M}_g$ is a morphism.
(*ii*) if $\mathcal{M}'_g$ is another scheme with a bijection $\psi' : S \to \mathcal{M}'_g$ such that (*i*) holds, then $\psi' \circ \psi^{-1} : \mathcal{M}_g \to \mathcal{M}'_g$ is a morphism.

We then call $\mathcal{M}_g$ a coarse moduli space of curves of genus g.

The moduli space defined above is less than perfect in the following sense. It would be very nice if there were a family of curves

$$\pi : \mathcal{U} \to \mathcal{M}_g,$$

such that for every family of curves $X \to B$ we would have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[dr] & \phi_{B,X}^*(\mathcal{U}) \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "\pi"] \\
& B \arrow[r, "\phi_{B,X}"] & \mathcal{M}_g
\end{tikzcd}$$

The family $\mathcal{U}$ would then be called a universal curve over $\mathcal{M}_g$ and $\mathcal{M}_g$ would be called a fine moduli space. Unfortunately, $\mathcal{M}_g$ does not exist as a fine moduli space (at least not if we require $\mathcal{M}_g$ to be a scheme). However, the coarse moduli space exists, and for $g > 2$ it contains a dense open subset which has a universal family.

Chapter 3
Tautological Rings

In Chapter 2 we defined the moduli space $\mathcal{M}_g$ of smooth surves of genus g over a field k. From now on, we shall always assume that k is algebraically closed and that g is at least 2.

Similarly, we may consider the moduli space of tuples $(C, p_1, \ldots, p_n)$, where C is a smooth curve of genus g over k and $p_1, \ldots, p_n$ are distinct points of C. Such tuples are called marked curves and the moduli space parametrizing them, denoted $\mathcal{M}_{g,n}$, is consequently called a moduli space of marked curves. The moduli space of curves marked with one point, $\mathcal{M}_{g,1}$, is given the symbol $\mathcal{C}_g$. There is a natural morphism $\pi : \mathcal{C}_g \to \mathcal{M}_g$ defined by "forgetting the point", i.e. $\pi([C,p]) = [C]$. Over the dense open subset, $\mathcal{M}_g^0$, of $\mathcal{M}_g$ consisting of curves without automorphisms, $\mathcal{C}_g$ is a universal curve. Therefore $\mathcal{C}_g$ is sometimes called, by abuse of language, the universal curve over $\mathcal{M}_g$.

$\mathcal{C}_g$ is a scheme over $\mathcal{M}_g$ and we may thus construct the n-fold fibre product of $\mathcal{C}_g$ over $\mathcal{M}_g$, denoted $\mathcal{C}_g^n$:

$$\mathcal{C}_g^n = \underbrace{\mathcal{C}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}_g}_{n \text{ times}}.$$

The space $\mathcal{C}_g^n$ parametrizes smooth curves marked with n, not necessarily distinct, points. For notational convenience we shall sometimes write $\mathcal{C}_g^1$ to mean $\mathcal{C}_g$ and $\mathcal{C}_g^0$ to mean $\mathcal{M}_g$.

For $m \geq n$ we have morphisms $\pi : \mathcal{C}_g^m \to \mathcal{C}_g^n$ defined by forgetting $m - n$ points. Especially important are the morphisms

$$\pi_{n,i} : \mathcal{C}_g^n \to \mathcal{C}_g^{n-1},$$

defined by forgetting the i:th point.

The spaces $\mathcal{C}_g^n$ have Chow rings $A(\mathcal{C}_g^n)$. These rings are however believed to be very big so one instead chooses to concentrate on a subring generated by the most important cycles. (Throughout, we shall work with rational coefficients in $A(\mathcal{C}_g^n)$ rather than integers).

Consider the morphism

$$\pi_1 : \mathcal{C}_g \to \mathcal{M}_g.$$

Denote by ω_{π_1} the sheaf of rational sections of $\mathrm{Coker}(d\pi_1 : \pi_1^* \Omega_{\mathcal{M}_g} \to \Omega_{\mathcal{C}_g})$. This sheaf is called the relative dualizing sheaf of π_1. Define K to be the first Chern class of ω_{π_1}, i.e.

$$K := c_1(\omega_{\pi_1}) \in A^1(\mathcal{C}_g).$$

We use K to define the so-called κ-classes

$$\kappa_i := \pi_{1*}(K^{i+1}) \in A^i(\mathcal{M}_g).$$

In particular, $\kappa_{-1} = 0$ and $\kappa_0 = 2g - 2$.

We may also define

$$\mathbb{E} := \pi_{1*}(\omega_{\pi_1}).$$

This is the Hodge bundle. It is a vector bundle of rank g on $\mathcal{M}_g$. Its fiber at each point $[C]$ of $\mathcal{M}_g$ is the space of holomorphic differentials on C. We define the λ-classes as

$$\lambda_i := c_i\left(\mathbb{E}\right) \in A^i(\mathcal{M}_g).$$

In particular, $\lambda_0 = 1$ and $\lambda_i = 0$ if $i > g$. The κ- and λ-classes generate a $\mathbb{Q}$-subalgebra of $A(\mathcal{M}_g)$. This subalgebra is in fact a graded ring which we call the tautological ring of $\mathcal{M}_g$ and denote by $R(\mathcal{M}_g)$. We write $R^i(\mathcal{M}_g)$ to denote the degree i component of $R(\mathcal{M}_g)$.

We shall define an analogous ring for $\mathcal{C}_g^n$. We therefore consider the morphism

$$\pi_{n,i} : \mathcal{C}_g^n \to \mathcal{C}_g^{n-1},$$

which forgets the i:th point. Let $\omega_{\pi_{n,i}}$ be its relative dualizing sheaf and define

$$K_i := c_1\left(\omega_{\pi_{n,i}}\right) \in A^1(\mathcal{C}_g^n).$$

We also have the class of points

$$[(C, p_1, \ldots, p_n)] \in \mathcal{C}_g^n,$$

such that $p_i = p_j$, $i \neq j$. This class is called a diagonal class and it is denoted by $D_{i,j}$. Finally, we pull back the κ- and λ-classes. By abuse of notation we shall also denote the pull-back of κ_i and λ_i by κ_i and λ_i, respectively. We now define the tautological ring of $\mathcal{C}_g^n$, denoted $R(\mathcal{C}_g^n)$, as the subalgebra generated by the K_i-, $D_{i,j}$-, κ- and λ-classes.

Remark 3.1. Note that to conform with the latter notation, the class K in $A^1(\mathcal{C}_g)$ should really be denoted by K_1. However, there is only one marked point to forget in $\mathcal{C}_g$ so there should be no confusion. We shall therefore drop the index whenever we work with $R(\mathcal{C}_g)$.

3.1 Known Results

An early result concerning the tautological ring is the following theorem of Mumford, [30].

Theorem 3.1 (Mumford). *The classes λ_i and κ_i are polynomials in the classes $\kappa_1, \dots, \kappa_{g-2}$.*

For instance, we have the following relation between the λ_i and the κ_j

$$\sum_{i=0}^{\infty} \lambda_i t^i = \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}\kappa_{2i-1}}{2i(2i-1)} t^{2i-1}\right),$$

where the B_{2i} are signed Bernoulli numbers.

A few years back, Mumfords result was improved quite a bit by Ionel, [23].

Theorem 3.2 (Ionel). *The $[g/3]$ classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate $R(\mathcal{M}_g)$.*

In the above theorem, $[x]$ denotes the integer closest to x.

We also have relations between other classes. A few important ones are the following, which were discovered by Harris and Mumford, [20].

Lemma 3.1 (Harris and Mumford). *The following identities hold in $R(\mathcal{C}_g^n)$:*

$$D_{i,n}D_{j,n} = D_{i,j}D_{i,n}, \qquad i < j < n, \tag{1}$$

$$D_{i,n}^2 = -K_i D_{i,n}, \qquad i < n, \tag{2}$$

$$K_n D_{i,n} = K_i D_{i,n}, \qquad i < n. \tag{3}$$

Using the above identities repeatedly, every monomial in the classes K_i and D_{ij} $(i < j < n)$ in $R(\mathcal{C}_g^n)$ can be rewritten as a monomial pulled back from $R(\mathcal{C}_g^{n-1})$ times either a single diagonal $D_{i,n}$ or a power of K_n.

Harris and Mumford also give the following formulas for the map $\pi_{n,n*} : R(\mathcal{C}_g^n) \to R(\mathcal{C}_g^{n-1})$.

Lemma 3.2 (Harris and Mumford). *If M is a monomial in $R(\mathcal{C}_g^n)$ which is pulled back from $R(\mathcal{C}_g^{n-1})$, then*

$$\pi_{n,n*}(M \cdot D_{i,n}) = M, \tag{1}$$

$$\pi_{n,n*}(M \cdot K_n^k) = M \cdot \pi^*(\kappa_{k-1}) = M \cdot \kappa_{k-1}. \tag{2}$$

Lemmas 3.1 and 3.2 will be very important in our calculations later but we will see their usefulness already in the proof of Corollary 3.1. The following vanishing result is due to Looijenga, [26].

Theorem 3.3 (Looijenga). *$R^j(\mathcal{C}_g^n) = 0$ if $j > g+n-2$ and $R^{g+n-2}(\mathcal{C}_g^n)$ is at most one-dimensional.*

Looijengas theorem was improved a bit by Faber, [11].

Theorem 3.4 (Faber). *κ_{g-2} is non-zero in $R(\mathcal{M}_g)$.*

It follows that $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. The non-vanishing of $R^{g+n-2}(\mathcal{C}_g^n)$ extends easily to higher n.

Corollary 3.1. *$R^{g+n-2}(\mathcal{C}_g^n)$ is one-dimensional.*

Proof. The monomial $D_{1,2} \cdot D_{1,3} \cdots D_{1,n} \cdot K_1^{g-1}$ in $R(\mathcal{C}_g^n)$ has degree $g+n-2$ and projects to κ_{g-2} in $R(\mathcal{M}_g)$. The class κ_{g-2} is non-zero by Theorem 3.4 so $D_{1,2} \cdots D_{1,n} K_1^{g-1}$ is non-zero. The result now follows from Theorem 3.3. □

3.2 The Faber Conjectures

We shall now state the so-called Faber conjectures. They were first stated in 1993 but not published until 1999, [10].

Conjecture 3.1 (Faber). *(a) The tautological ring $R^*(\mathcal{M}_g)$ is Gorenstein with socle in degree $g-2$, i.e. it vanishes in degrees $> g-2$, is 1-dimensional in degree $g-2$ and, when an isomorphism $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$ is fixed, the natural pairing,*

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q},$$

is perfect.

(b) The $[g/3]$ classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate $R^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.

(c) There exist explicit formulas for the proportionalities in degree $g-2$, which may be given as follows. Let $(d_1, \ldots, d_k)$ be a partition of $g-2$ into positive integers. We define expressions $\langle \tau_{d_1+1}\tau_{d_2+1}\cdots\tau_{d_k+1}\rangle \in R^{g-2}(\mathcal{M}_g)$ in two ways,

$$\langle \tau_{d_1+1}\tau_{d_2+1}\cdots\tau_{d_k+1}\rangle = \frac{(2g-3+k)!(2g-1)!!}{(2g-1)!\prod_{j=1}^{k}(2d_j+1)!!}\kappa_{g-2}, \tag{1}$$

$$\langle \tau_{d_1+1}\tau_{d_2+1}\cdots\tau_{d_k+1}\rangle = \sum_{\sigma \in \mathfrak{S}_n} \kappa_\sigma, \tag{2}$$

where $\kappa_\sigma = \kappa_{|\alpha_1|}\kappa_{|\alpha_2|\cdots\kappa_{|\nu(\sigma)|}}$ for a decomposition $\sigma = \alpha_1\alpha_2\cdots\alpha_{\nu(\sigma)}$ into disjoint cycles, including the 1-cycles. $|\alpha|$ is defined as the sum of the elements in the cycle α, where we think of $\mathfrak{S}_k$ as acting on the k-tuples with entries $d_1, d_2, \ldots, d_k$. (1) and (2) allow us to express every monomial κ_I of degree $g-2$ (where I is a multi-index) as a multiple of κ_{g-2}.

Above, $n!!$ denotes the double factorial, defined for non-negative odd integers $n = 2k-1$ as $n!! = \prod_{i=1}^{k}(2i-1)$.

It is worth mentioning that Theorems 3.3 and 3.4 together prove the first half of part (a) of the conjecture and that Theorem 3.2 proves part (b). We also have the following result of Liu and Xu, [24].

Theorem 3.5 (Liu and Xu). *Conjecture 3.1. (c) is true.*

Another proof of this result has later been given by Buryak and Shadrin, [6]. Consider the projection $\pi = \pi_{n+1,n+1} : \mathcal{C}_g^{n+1} \to \mathcal{C}_g^n$ that forgets the $(n+1)$:st point and let Δ_{n+1} denote the sum,

$$\Delta_{n+1} = D_{1,n+1} + D_{2,n+1} + \cdots + D_{n,n+1},$$

as well as the corresponding divisor. Denote by ω_i the line bundle on $\mathcal{C}_g^n$ obtained by pulling back ω_{π_1} along the projection $\pi_i : \mathcal{C}_g^n \to \mathcal{C}_g$ onto the i:th factor. Define a coherent sheaf $\mathbb{F}_n$ on $\mathcal{C}_g^n$ by

$$\mathbb{F}_n = \pi_*\left(\mathcal{O}_{\Delta_{n+1}} \otimes \omega_{n+1}\right).$$

We may now state Faber's third conjecture, [10].

Conjecture 3.2 (Faber). *Let $I_g \subset \mathbb{Q}[\kappa_1, \ldots, \kappa_{g-2}]$ be the ideal generated by the polynomials of the form,*

$$\pi_*(M \cdot c_j(\mathbb{F}_{2g-1} - \mathbb{E})),$$

with $j \geq g$ and M a monomial in the K_i and $D_{i,j}$ and $\pi : \mathcal{C}_g^{2g-1} \to \mathcal{M}_g$ the forgetful map. Then $\mathbb{Q}[\kappa_1, \ldots, \kappa_{g-2}]/I_g$ is Gorenstein with socle in degree $g-2$. Hence, it is isomorphic to $R(\mathcal{M}_g)$.

It should be mentioned that the above conjecture is based on the following result, [10].

Theorem 3.6 (Faber). *If $n \geq 2g-1$ and $j \geq n-g+1$, then $c_j(\mathbb{F}_n - \mathbb{E}) = 0$.*

In [10], Faber uses Theorem 3.6 to obtain an algorithm to compute relations in $R(\mathcal{M}_g)$. An important step is the following identity

$$c(\mathbb{F}_d) = (1+K_1)(1+K_2-\Delta_2)(1+K_3-\Delta_3)\cdots(1+K_d-\Delta_d).$$

We also have the following identity of Mumford, [30]

$$c(\mathbb{E})^{-1} = c(\mathbb{E}^\vee) = \sum_{i=0}^{g} (-1)^i \lambda_i = 1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g.$$

The idea is now to choose n and j so that Theorem 3.6 is satisfied, so that $c_j(\mathbb{F}_n - \mathbb{E}) = 0$. If we multiply this equation by a monomial M in the K_i's and $D_{i,j}$'s we see that $M \cdot c_j(\mathbb{F}_n - \mathbb{E}) = 0$. We may then push this relation down one step at a time using the maps $\pi_{d,d*}$ to obtain a relation in $R(\mathcal{M}_g)$. To perform the calculations one may use Lemma 3.1 and 3.2. In this way we may calculate a number of relations in $R^i(\mathcal{M}_g)$ and thus obtain an upper bound of its dimension.

We shall now discuss how to obtain a lower bound for the dimension. First we remind the reader of the following definition.

Definition 3.1. Let $\kappa_I = \kappa_{i_1}^{n_1} \cdot \kappa_{i_2}^{n_2} \cdots \kappa_{i_r}^{n_r}$ be a monomial in the κ-classes. Then the degree of κ_I is

$$\deg(\kappa_I) = \sum_{j=1}^{r} n_j i_j.$$

Let κ_I be a monomial in the κ-classes of degree i and let κ_J be a monomial in the κ-classes of degree j, where $j = g - 2 - i$. Then $\kappa_I \cdot \kappa_J$ is a monomial of degree $g-2$. Since $R^{g-2}(\mathcal{M}_g)$ is one-dimensional and generated by κ_{g-2}, we may express $\kappa_I \cdot \kappa_J$ as a rational multiple of κ_{g-2}, $\kappa_I \cdot \kappa_J = r \cdot \kappa_{g-2}$. We therefore make the following definition.

Definition 3.2. Let κ_I be a monomial of degree $g-2$ in the κ-classes. Define $r(\kappa_I)$ to be the rational number which satisfies

$$\kappa_I = r(\kappa_I) \cdot \kappa_{g-2}.$$

We remark that Theorem 3.5 may be used to calculate the numbers $r(\kappa_I)$.

From this point on we fix a monomial ordering $<_\kappa$ of the monomials in the κ-classes. Which one is of no importance so the reader may think of his or her favourite.

Recall that the partition function, p, is the function which for each non-negative integer gives the number of ways of writing it as an unordered sum of positive integers. For instance, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$ and $p(4) = 5$. Since it is not completely uncommon to define the partition function only for positive integers, we point out that $p(0) = 1$ (the empty partition).

Definition 3.3. Let $i \leq g-2$ be a non-negative integer. We define the $p(i) \times p(g-2-i)$-matrix $P_{g,i}$ as follows. Let κ_K be the kth monic monomial of degree i and let κ_L be the lth monic monomial of degree $g-2-i$ (according to $<_\kappa$). Then the (k,l)th entry of $P_{g,i}$ is $r(\kappa_K \cdot \kappa_L)$. We shall refer to matrices of this type as pairing matrices.

The monomials κ_I, where I is a multi-index such that

$$\sum_{i_r \in I} r \cdot i_r = i,$$

generate $R^i(\mathcal{M}_g)$ by Theorem 3.1. Note that every $\mathbb{Q}$-linear relation among the monomials κ_I of degree i clearly gives a linear relation among the rows of $P_{g,i}$. Hence, if the rank of $P_{g,i}$ is n, then $R^i(\mathcal{M}_g)$ has dimension at least n.

We now have a two-step program to prove Conjecture 3.3 close at hand. First we compute the rank of $P_{g,i}$ to obtain a lower bound n on the dimension of $R^i(\mathcal{M}_g)$. We then multiply the relation in Theorem 3.6 by monomials and use Lemmas 3.1 and 3.2 to push these relations to $R(\mathcal{M}_g)$. We then pick out the degree i part of the relation which must be a relation in $R^i(\mathcal{M}_g)$. If we can find enough relations we may obtain n as an upper bound for the dimension. Then we are done. This idea is due to Faber, [10], and has been used by him to prove Conjecture 3.2 for $g \leq 23$.

Chapter 4
The Tautological Ring of the Universal Curve

The aim of this project has been to pose and verify questions similar to Conjecture 3.1 (a) and Conjecture 3.3 in the tautological ring of the universal curve, $R(\mathcal{C}_g)$. The analogue of Conjecture 3.1 (a) is the following.

Question 4.1 *Is the tautological ring $R(\mathcal{C}_g)$ Gorenstein with socle in degree $g-1$, i.e. does it vanish in degrees $> g-1$, is it one dimensional in degree $g-1$ and, when an isomorphism $R^{g-1}(\mathcal{C}_g) \cong \mathbb{Q}$ is fixed, is the natural pairing,*

$$R^i(\mathcal{C}_g) \times R^{g-1-i}(\mathcal{C}_g) \to R^{g-1}(\mathcal{C}_g) \cong \mathbb{Q},$$

perfect?

We note that the first part follows from Theorem 3.3 and Corollary 3.1. What remains is to check whether the pairing is perfect or not.

We shall now state the analogue of Conjecture 3.3.

Question 4.2 *In the polynomial ring $\mathbb{Q}[K, \kappa_1, \ldots, \kappa_{g-2}]$, let J_g be the ideal generated by relations of the form,*

$$\pi_*(M \cdot c_j(\mathbb{F}_{2g-1} - \mathbb{E})),$$

with $j \geq g$ and M a monomial in the K_i and D_{ij}, and $\pi : \mathcal{C}_g^{2g-1} \to \mathcal{C}_g$ a forgetful map. Is the quotient ring $\mathbb{Q}[K, \kappa_1, \ldots, \kappa_{g-2}]/J_g$ is Gorenstein with socle in degree $g-1$? If so, it is isomorphic to the tautological ring $R^*(\mathcal{C}_g)$.

Our strategy will be to adapt the technique of Faber described in Chapter 3 to $R(\mathcal{C}_g)$. To do so we first note that we may stop pushing down at $R(\mathcal{C}_g)$ instead of at $R(\mathcal{M}_g)$. Thus, the method for generating relations extends to $R(\mathcal{C}_g)$ without any trouble.

In question 4.2 it would be natural to consider the ideal J'_g, generated by the ideal J_g and the relations pulled back from $R(\mathcal{M}_g)$ instead of the ideal J_g. Of course, the pulled back relations must also be valid but it has turned out in the calculations for low genera that J_g is the full set of relations, i.e. that $J_g = J'_g$. Hence, at this point there is little reason to pose question 4.2 differently. However, it would not be surprising if it would be the case that J_g and J'_g are different for higher genera than those that has been investigated this far.

4.1 Pairing Matrices

Recall the matrices $P_{g,i}$, introduced in Definition 3.3. There are also pairing matrices corresponding to the matrices $P_{g,i}$ on $R(\mathcal{C}_g)$. To define these we need a bit of preparation.

In $R(\mathcal{C}_g)$ we only have one more class than in $R(\mathcal{M}_g)$, namely the class $K_1 = K$. Hence, Theorem 3.1 gives that $R(\mathcal{C}_g)$ is generated by the monomials in $\kappa_1, \ldots, \kappa_{g-2}$ and K. K has degree 1 so a monomial $M = K^j \kappa_1^{n_1} \cdots \kappa_{g-2}^{n_{g-2}}$ has degree

$$\deg(M) = j + \sum_{j=1}^{g-2} n_j \cdot j.$$

We define a monomial ordering as follows. First fix the same monomial ordering $<_\kappa$ as in Chapter 3 on the monomials in the κ-classes. We then note that any monomial M in the κ-classes and K can be written as $M = K^j \cdot \kappa_I$ where κ_I is a monomial in the κ-classes only. We now define a monomial order as follows.

Definition 4.1. Let $M = K^r \kappa_I$ and $N = K^s \kappa_J$ be monomials in the κ-classes and K. We define a monomial ordering $<_*$ by

(a) $M <_* N$ if $r < s$ or,
(b) $M <_* N$ if $r = s$ and $\kappa_I <_\kappa \kappa_J$.

Note that by Theorem 3.3 and Corollary 3.1 any monomial, M, of degree $g-1$ is a rational multiple of $K\kappa_{g-2}$, i.e. $M = s(M) \cdot K\kappa_{g-2}$. Using Lemma 3.2 we have that $\pi_{1,1*}(M) = s(M) \cdot \kappa_{g-2}$. We use this observation to define the analogues of the matrices $P_{g,i}$.

Definition 4.2. Let M be the kth monic monomial of degree i according to $<_*$ and let N be the lth monic monomial of degree $(g-1-i)$ according to $<_*$. Define $s_{k,l}$ as the rational number satisfying $\pi_{1,1*}(M \cdot N) = s_{k,l}\kappa_{g-2}$. We define the matrix $Q_{g,i}$ as

$$Q_{g,i} = (s_{k,l}).$$

The dimensions of $Q_{g,i}$ are

$$\left(\sum_{r=0}^{i} p(r)\right) \times \left(\sum_{r=0}^{g-1-i} p(r)\right),$$

where p is the partition function (remember that we have defined $p(0) = 1$). As with the matrices $P_{g,i}$, the rank of $Q_{g,i}$ determines a lower bound for the dimension of $R^i(\mathcal{C}_g)$.

The definition of $Q_{g,i}$ suggests that there should be a relationship between $Q_{g,i}$ and the matrices $P_{g,j}$. To provide this relationship it is convenient to introduce some notation. To this end we recall that the (i,j)th entry of $P_{g,i}$ corresponds to the monomial $\kappa_I\ \kappa_J$, where κ_I is the ith monomial of degree i with respect to the chosen monomial ordering, and κ_J is the jth monomial of degree $(g-2-i)$. Hence, we may think of ith row of $P_{g,i}$ as being labelled by the monomial κ_I.

Definition 4.3. Let $j \leq i$ be a positive integer. Define $P_{g,i}^j$ as the $p(i-j) \times p(g-2-i)$-submatrix of $P_{g,i}$ consisting of the rows of $P_{g,i}$ which are labelled by monomials κ_I containing at least one factor κ_j.

It turns out to be notationally convenient to define,

$$P_{g,i}^0 = (2g-2) \cdot P_{g,i},$$

and,

$$P_{g,i}^{-1} = \text{ the zero matrix of size } p(i+1) \times p(g-2-i).$$

We are now ready to state the following Proposition.

Proposition 4.1. *(a) Let $Q_{g,i}$ and $P_{g,j}^r$ be defined as above and let $i \geq 1$. Then,*

$$Q_{g,i} = \begin{pmatrix} P_{g,i-1}^{-1} & P_{g,i}^0 & P_{g,i+1}^1 & P_{g,i+2}^2 & \cdots & P_{g,g-2}^{g-2-i} \\ P_{g,i-1}^0 & P_{g,i}^1 & P_{g,i+1}^2 & \cdots & \cdots & \vdots \\ P_{g,i-1}^1 & P_{g,i}^2 & \ddots & & & \vdots \\ P_{g,i-1}^2 & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ P_{g,i-1}^{i-1} & \cdots & \cdots & \cdots & \cdots & P_{g,g-2}^{g-2} \end{pmatrix}.$$

(b) The rank of $Q_{g,0}$ is 1.

Proof. (a) Denote the monomial labelling the rth row of $Q_{g,i}$ by N_r and the monomial labelling the sth column of $Q_{g,i}$ by N_s.

Consider first the submatrix of $Q_{g,i}$ corresponding to rows and columns labelled by monomials N_r and N_s not containing a factor K. Then $N_r \cdot N_s$ projects to 0 so this submatrix consists entirely of zeros. With the above notation, this submatrix is equal to $P_{g,i-1}^{-1}$.

Now consider a submatrix C of $Q_{g,i}$ corresponding to rows and columns labelled by monomials N_r and N_s such that ,

i) $N_r = K^{n_r} N_r'$ and $N_s = K^{n_s} N_s'$ where K does not divide N_r' or N_s' and, *ii)* not both n_r and n_s are zero.

Then,

$$\pi_{1,1}(N_r \cdot N_s) = \pi_{1,1}(K^{n_r+n_s} \cdot N'_r \cdot N'_s) = \kappa_{n_r+n_s-1} \cdot N'_r \cdot N'_s.$$

Note that $\kappa_{n_r+n_s-1} \cdot n'_r$ is a monomial in the κ_i's of degree $i+n_s-1$ containing a factor $\kappa_{n_r+n_s-1}$ and that N'_s is a monomial of degree $g-1-i-n_s = g-2-(i+n_s-1)$ in the κ_i's. Further, every monomial in the κ_i's of degree $i+n_s-1$ containing a factor $\kappa_{n_r+n_s-1}$ is the image of some monomial $K^{n_r+n_s} \cdot N'_r$ and every polynomial of degree $g-2-(i+n_s-1)$ is represented by the N'_s's. By our choice of monomial order labelling the rows and columns of $Q_{g,i}$ we now see that $C = P^{n_r+n_s-1}_{g,i+n_s-1}$. This completes the proof of (a).

(b) The only row of $Q_{g,0}$ is labelled by 1. The last column of $Q_{g,0}$ is labelled by K^{g-1}. Hence, the $(1, g-1)$:st entry of $Q_{g,0}$ is $1 \neq 0$. Hence, $Q_{g,0}$ must have rank 1. □

The merit of Proposition 4.1 is that it tells us how to compute the matrices $Q_{g,i}$ without having to project monomials of $R^*(\mathcal{C}_g)$ down to $R^*(\mathcal{M}_g)$. Hence, we have reduced the problem of computing the matrices $Q_{g,i}$ to computing the matrices $P_{g,i}$, which are smaller and easier to compute. We shall describe a rather efficient way of doing this shortly. However, first we note a few things which reduce the calculations a bit.

Firstly, the (k,l):th element of $Q_{g,i}$ is $s_{k,l}$, where $s_{k,l}$ is the rational number satisfying $\pi_{1,1*}(M \cdot N) = s_{k,l}\kappa_{g-2}$, where M is the kth monic monomial of degree i and N is the l:th monic monomial of degree $g-1-i$. But since $M \cdot N = N \cdot M$ we also have that $\pi_{1,1*}(N \cdot M) = s_{k,l}\kappa_{g-2}$. Hence, $s_{k,l} = s'_{l,k}$, where $s'_{l,k}$ is the rational number satisfying $\pi_{1,1*}(N \cdot M) = s'_{l,k}\kappa_{g-2}$. We thus have that $Q_{g,g-1-i} = Q^T_{g,i}$. Similarly, $P_{g,g-2-i} = P^T_{g,i}$. Hence, we only have to compute $P_{g,i}$ for $i \leq [(g-2)/2]$ in order to apply Proposition 4.1 and we only have to compute the rank of $Q_{g,i}$ for $i \leq [(g-1)/2]$ since $\mathrm{rank}(Q_{g,g-1-i}) = \mathrm{rank}(Q_{g,i})$.

Our second remark is less straightforward. There is a homomorphism from the Chow ring of $\mathcal{M}_g$, $A(\mathcal{M}_g)$, (with rational coefficients) to the rational cohomology ring of $\mathcal{M}_g$, $H^*(\mathcal{M}_g, \mathbb{Q})$. We denote the image of κ_i by $\hat{\kappa}_i$. The following theorem of Madsen and Weiss, [29] (known as "the Mumford conjecture" or "the Madsen-Weiss theorem") gives quite a bit of information about $H^*(\mathcal{M}_g, \mathbb{Q})$.

Theorem 4.1 (Madsen and Weiss). *Let $\mathbb{Q}[x_1, x_2, x_3, \ldots]$ be the polynomial ring over $\mathbb{Q}$ in infinitely many variables x_i, where x_i has degree $2i$. Then, the map*

$$\mathbb{Q}[x_1, x_2, x_3, \ldots] \to H^*(\mathcal{M}_g, \mathbb{Q})$$

sending $x_i \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$ to $\hat{\kappa}_i \in H^{2i}(\mathcal{M}_g, \mathbb{Q})$ is an isomorphism in degrees less than $(g-1)/2$.

It should probably be mentioned that the Madsen-Weiss theorem is usually stated a bit differently. The above form can be found in the article [9].

We also have the following stability result of Boldsen, [5].

Theorem 4.2 (Boldsen). *If $g \equiv 1$ or $2 \mod 3$ and $i \leq 2[g/3]$, then $H^i(\mathcal{M}_g)$ is independent of g. If $g \equiv 0 \mod 3$ and $i < 2g/3$, then $H^i(\mathcal{M}_g)$ is independent of g.*

The result below follows.

Theorem 4.3. *In $R(\mathcal{M}_g)$, there are no relations in codimensions $< g/3$.*

By a result of Looijenga, [27], we also have the following.

Theorem 4.4. *In $R(\mathcal{C}_g)$, there are no relations in codimensions $< g/3$.*

Hence, we do not have to compute the rank of $Q_{g,i}$ for $i < [g/3]$. What needs to be computed is the rank of $Q_{g,i}$ for $[g/3] \leq i \leq [g/2]$. This is done by means of Proposition 4.1 and the following algorithm of Liu and Xu, [25].

4.2 Computing $P_{g,i}$

In this section we describe an algorithm due to Liu and Xu, [25], by means of which one may efficiently compute the matrices $P_{g,i}$.

Let $\mathbf{m} = (m_1, m_1, \cdots)$ be a sequence of non-negative integers with only finitely many of the m_i non-zero. The set of such sequences is a monoid under coordinatewise addition. Define

$$|\mathbf{m}| = \sum_{i=1}^{\infty} i \cdot m_i, \quad ||\mathbf{m}|| = \sum_{i=1}^{\infty} m_i, \quad \mathbf{m}! = \prod_{i=1}^{\infty} m_i!.$$

A sequence $\mathbf{m}$ determines a monomial, $\kappa_{\mathbf{m}}$, in the κ-classes as follows.

$$\kappa_{\mathbf{m}} = \prod_{i=1}^{m} \kappa_i^{m_i}$$

We inductively define constants $\beta_{\mathbf{m}}$ and by setting $\beta_{\mathbf{0}} = 1$ and requiring

$$\sum_{\mathbf{m}'+\mathbf{m}''=\mathbf{m}} \frac{(-1)^{||\mathbf{m}'||}\beta_{\mathbf{m}'}}{\mathbf{m}''!(2|\mathbf{m}''|+1)!!} = 0 \quad \text{when } \mathbf{m} \neq \mathbf{0}.$$

We also define constants $\gamma_{\mathbf{m}}$ as

$$\gamma_{\mathbf{m}} = \frac{(-1)^{||\mathbf{m}||}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}.$$

Note that Liu and Xu writes $\beta_{\mathbf{m}}^{-1}$ instead of $\gamma_{\mathbf{m}}$. This is however misleading since $\beta_{\mathbf{m}} \cdot \gamma_{\mathbf{m}} \neq 1$ in most cases. We use $\beta_{\mathbf{m}}$ and $\gamma_{\mathbf{m}}$ to define new constants, $C_{\mathbf{m}}$.

$$C_{\mathbf{m}} = \sum_{\mathbf{m}'+\mathbf{m}''=\mathbf{m}} 2|\mathbf{m}'|\beta_{\mathbf{m}'}\gamma_{\mathbf{m}''}.$$

Now let $|\mathbf{m}| \leq g-2$ and define further constants $F_g(\mathbf{m})$ via

$$|\mathbf{m}| \cdot F_g(\mathbf{m}) = (g-1) \cdot \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{m}'\neq 0}} C_{\mathbf{m}'} F_g(\mathbf{m}''),$$

and $F_g(\mathbf{0}) = 1$. We now have the following result of Liu and Xu, [25].

Lemma 4.1 (Liu and Xu). *Let $|\boldsymbol{m}| = g-2$ and $r(\kappa_{\boldsymbol{m}})$ as defined in Definition 3.2. Then $r(\kappa_{\boldsymbol{m}})$ is given by*

$$r(\kappa_{\boldsymbol{m}}) = \frac{(2g-3)!! \cdot \boldsymbol{m}!}{2g-2} \cdot F_g(\boldsymbol{m}).$$

Lemma 4.1 gives a much more efficient way to compute $P_{g,i}$, especially if one wants to compute many different $P_{g,i}$, since much of the work can be reused.

Example 4.1. Since the definitions are quite involved it might be nice to see an example. We shall therefore compute $r(\kappa_{(2,0,\cdots)})$ for $g = 4$.

First take $\mathbf{m} = (1, 0, 0, \cdots)$. Then

$$\begin{aligned} 0 &= \frac{(-1)^{||\mathbf{0}||}\beta_{\mathbf{0}}}{(1,0,0,\cdots)!(2|(1,0,0,\cdots)|+1)!!} + \frac{(-1)^{||(1,0,0,\cdots)||}\beta_{(1,0,0,\cdots)}}{\mathbf{0}!(2|\mathbf{0}|+1)!!} = \\ &= \frac{1\cdot 1}{1\cdot(2\cdot 1+1)!!} - \frac{\beta_{(1,0,0,\cdots)}}{1\cdot(2\cdot 0+1)!!} = \\ &= \frac{1}{3} - \beta_{(1,0,0,\cdots)}. \end{aligned}$$

Hence, $\beta_{(1,0,0,\cdots)} = \frac{1}{3}$. We continue with $\mathbf{m} = (2, 0, 0, \cdots)$. Then

$$\begin{aligned} 0 &= \frac{(-1)^{||\mathbf{0}||}\beta_{\mathbf{0}}}{(2,0,\cdots)!(2|(2,0,\cdots)|+1)!!} + \frac{(-1)^{||(1,0,\cdots)||}\beta_{(1,0,\cdots)}}{(1,0,\cdots)!(2|(1,0,\cdots)|+1)!!} + \\ &+ \frac{(-1)^{||(2,0,\cdots)||}\beta_{(2,0,\cdots)}}{\mathbf{0}!(2|\mathbf{0}|+1)!!} = \\ &= \frac{1\cdot 1}{2\cdot(2\cdot 2+1)!!} - \frac{1\cdot\frac{1}{3}}{1\cdot(2\cdot 1+1)!!} + \frac{1\cdot\beta_{(2,0,\cdots)}}{1\cdot(2\cdot 0+1)!!} = \\ &= \frac{1}{30} - \frac{1}{9} + \beta_{(2,0,\cdots)}. \end{aligned}$$

We thus obtain that $\beta_{(2,0,\cdots)} = \frac{7}{90}$. We also compute the corresponding $\gamma_{\mathbf{m}}$:

$$\gamma_{\mathbf{0}} = \frac{(-1)^{||\mathbf{0}||}}{\mathbf{0}!(2|\mathbf{0}|+1)!!} = 1,$$

$$\gamma_{(1,0,\cdots)} = \frac{(-1)^{||(1,0,\cdots)||}}{(1,0,\cdots)!(2|(1,0,\cdots)|+1)!!} = -\frac{1}{3},$$

and

$$\gamma_{(2,0,\cdots)} = \frac{(-1)^{||(2,0,\cdots)||}}{(2,0,\cdots)!(2|(2,0,\cdots)|+1)!!} = \frac{1}{30}.$$

We continue by computing the $C_{\mathbf{m}}$:

$$C_{(1,0,\cdots)} = 2|(1,0,\cdots)|\beta_{(1,0,\cdots)}\gamma_{\mathbf{0}} + 2|\mathbf{0}|\beta_{\mathbf{0}}\gamma_{(1,0,\cdots)} = 2\cdot 1\cdot\frac{1}{3}\cdot 1 = \frac{2}{3},$$

and

$$\begin{aligned} C_{(2,0,\cdots)} &= 2|(2,0,\cdots)|\beta_{(2,0,\cdots)}\gamma_{\mathbf{0}} + 2|(1,0,\cdots)|\beta_{(1,0,\cdots)}\gamma_{(1,0,\cdots)} = \\ &= 2\cdot 2\cdot\frac{7}{90}\cdot 1 + 2\cdot 1\cdot\frac{1}{3}\cdot\left(-\frac{1}{3}\right) = \\ &= \frac{28}{90} - \frac{2}{9} = \\ &= \frac{4}{45}. \end{aligned}$$

Up to now, everything holds for all $g \geq 2$. However, $F_g(\mathbf{m})$ depends on g. We therefore choose $g = 4$.

$$|(1,0,\cdots)|F_4((1,0,\cdots)) = (4-1)\cdot\frac{2}{3}\cdot 1.$$

We conclude that $F_4((1,0,\cdots)) = 2$. We also have

$$|(2,0,\cdots)|F_4((2,0,\cdots)) = (4-1)\cdot\left(\frac{4}{45} + \frac{2}{3}\cdot 2\right) = \frac{64}{15}.$$

Hence, $F_4((2,0,\cdots)) = \frac{32}{15}$. Lemma 4.1 now gives that

$$r(\kappa_{(2,0,\cdots)}) = \frac{(2\cdot 4-3)!!\cdot(2,0,\cdots)!}{2\cdot 4-2}\cdot\frac{32}{15} = \frac{15\cdot 2}{6}\cdot\frac{32}{15} = \frac{32}{3}.$$

Since $\kappa_{(2,0,\cdots)} = \kappa_1^2$, this is another way of expressing that in $R^2(\mathcal{M}_4)$, the relation

$$\kappa_1^2 = \frac{32}{3}\cdot\kappa_2,$$

holds. This relation can also be found in [10].

4.3 The Rank of $Q_{g,i}$

I have used Proposition 4.1 and Lemma 4.1 to write a Maple[1] program for computing the rank of $Q_{g,i}$. The results for $g \leq 27$ are shown in Table 4.1 below.

Write $g = 3k - l - 1$ with k a positive integer and l a non-negative integer. In [10], Faber remarked that the computational evidence suggests that the number of relations in $R^k(\mathcal{M}_g)$ only depends on l as long as $2k \leq g-2$. Under this assumption, $a(l)$ is defined to be the number of relations in $R^k(\mathcal{M}_g)$. $a(l)$ has been computed in [10] for $0 \leq l \leq 9$. This has later been extended to $l \leq 14$ in [25]. We show the results for $0 \leq l \leq 11$ in Table 4.2.

Faber and Zagier have guessed that $a(l)$ equals the number of partitions of l without any parts other than 2 which are congruent to 2 modulo 3. The guess is supported by the following (see also [1]). Let $\mathbf{p} = \{p_1, p_3, p_4, p_6, p_7, p_8, p_9, \ldots\}$ be a collection of variables indexed by the positive integers not congruent to 2 modulo 5. Define

$$\Psi(t, \mathbf{p}) = \sum_{i=0}^{\infty} t^i p_{3i} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} t^j + \sum_{i=0}^{\infty} t^i p_{3i+1} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} \frac{6j+1}{6j-1} t^j,$$

where we take $p_0 = 1$. Let $\sigma = (\alpha_1, 0, \alpha_3, \alpha_4, 0, \alpha_6 \ldots)$ be a sequence of non-negative integers with all coordinates with indices congruent to 2 modulo 5 equal to zero. Define

$$\mathbf{p}^\sigma = p_1^{\alpha_1} p_3^{\alpha_3} p_4^{\alpha_4} \cdots .$$

Define constants $C_r(\sigma)$ via

$$\log\left(\Psi(t, \mathbf{p})\right) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r(\sigma) t^r \mathbf{p}^\sigma .$$

We use these constants to define

$$\gamma = \sum_{\sigma} \sum_{i=0}^{\infty} C_r(\sigma) \kappa_r t^r \mathbf{p}^\sigma .$$

It was show by Faber and Zagier that the relation

$$\left[\exp\left(-\gamma\right)\right]_{t^r \mathbf{p}^\sigma} = 0,$$

holds in the Gorenstein quotient of $R(\mathcal{M}_g)$ when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \mod 2$. These are the so-called FZ-relations. It was shown quite recently by Pandharipande and Pixton, [31], that these relations also hold in $R(\mathcal{M}_g)$. These relations are sufficiently many for codimensions $\leq \lfloor (g-2)/2 \rfloor$, but it is not clear whether these relations are linearly independent or not.

[1] Maple©is a trademark of Waterloo Maple Inc.

Note the central role of positive integers not congruent to 2 modulo 3 in the above. This phenomenon will reappear elsewhere in a moment.

With this in mind, it might be interesting to investigate whether a similar behaviour can be observed in $R(\mathcal{C}_g)$. We therefore note that the expected number of relations, n, in codimension k is given through the formula

$$n = \sum_{i=0}^{k} p(i) - \text{rank}(Q_{g,k}).$$

Here $p(i)$ is the partition function extended with $p(0) = 1$. The computations for $l \leq 9$ suggested that the number of relations is a function of l only, as long as $2k \leq g-1$. Even though this turned out not to be the case, we shall momentarily pretend that n is a function of l. We show the computations of n for $0 \leq l \leq 11$ in Table 4.3.

Using Table 4.3, Faber and I each guessed a formula for n as a function of l (although at that time I had only computed the values for $g \leq 26$). Faber's guess, b_F, was

$$b_F(l) = \sum_{\substack{i=0 \\ i \not\equiv 2 \ (\mathrm{mod}\ 3)}}^{l} a(l-i),$$

where a is the a-function discussed above. My guess was the following recursive formula

$$b_B(l) = 2\sum_{i=0}^{l-1} a(i) + a(l) - b_B(l-1) - b_B(l-2), \quad l \geq 2,$$

with initial values $b_B(0) = a(0)$ and $b_B(1) = a(0)+a(1)$. However, the guesses are only superficially different.

Proposition 4.2. *Define b_F by*

$$b_F(l) = \sum_{\substack{i=0 \\ i \not\equiv 2 \ (\mathrm{mod}\ 3)}}^{l} a(l-i),$$

and b_B by the recursion

$$b_B(l) = 2\sum_{i=0}^{l-1} a(i) + a(l) - b_B(l-1) - b_B(l-2), \quad l \geq 2,$$

with initial values $b_B(0) = a(0)$ and $b_B(1) = a(0) + a(1)$. Then $b_F = b_B$.

Proof. The statement is evidently true for $l = 0$ and $l = 1$. Suppose that it is true for all $i < l$ for some $l > 1$. We have

$$\begin{aligned}
b_B(l) &= 2\sum_{i=0}^{l-1} a(i) + a(l) - b_B(l-1) - b_B(l-2) = \\
&= 2\sum_{i=0}^{l-1} a(i) + a(l) - \left(2\sum_{i=0}^{l-2} a(i) + a(l-1) - b_B(l-2) - b_B(l-3)\right) - \\
&- b_B(l-2) = \\
&= a(l-1) + a(l) + b(l-3).
\end{aligned}$$

But, by assumption, $b_B(l-3) = b_F(l-3)$. Hence

$$b_B(l) = a(l) + a(l-1) + \sum_{\substack{i=0 \\ i\not\equiv 2 \pmod 3}}^{l-3} a(l-i) = \sum_{\substack{i=0 \\ i\not\equiv 2 \pmod 3}}^{l} a(l-i) = b_F(l).$$

The result now follows by induction. □

From now on, we write b_G (G for guess) to denote the function $b_F = b_B$.

Our guess, b_G, gives the right number of relations n when $0 \le l \le 9$ but it gives the value $b_G(10) = 90$ instead of the value $n = 91$ which was obtained by computing the rank of $Q_{25,12}$. To investigate the matter further I computed the rank of $Q_{28,13}$ and $Q_{31,14}$. Both computations gave the predicted value $n = b_G(10) = 90$ which suggests that $Q_{25,12}$ is exceptional. Noteworthy is that the anomaly occurs in the middle dimension, $(g-1)/2$.

The above results suggest that n may exhibit a similar behaviour in the middle dimension also for $g > 25$. If this is so, we expect an anomaly for $g = 27$, $k = 13$. The rank of $Q_{27,13}$ gives $n = 120$ while $b_G(11) = 119$. Computing the rank of $Q_{30,14}$ again yields the predicted value, $n = b_G(11) = 119$.

One way to avoid this anomaly would be to require $2k \le g - 2$ instead of $2k \le g - 1$, although this is not very appealing (and very ad hoc). It might be interesting to mention in this context that the method of Faber has been unsuccessful in proving the Faber conjectures in $R^{12}(\mathcal{M}_{24})$. Note that also here the problem arises in the middle dimension.

4.4 Generating Relations

We earlier described a method for generating relations. Even though the method is rather easy in principle, its computational complexity is quite an obstacle. We shall discuss a few tricks which have helped to make the computations more efficient.

The first step of the algorithm is to pick a monomial M in $R(\mathcal{C}_g^{2g-1})$ in the K and $D_{i,j}$-classes. However, the set of all such polynomials is much too large already for low g. The computations so far suggest that the algorithm described below produces enough relations.

$g \setminus i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
2	1	1																									
3	1	2	1																								
4	1	2	2	1																							
5	1	2	3	2	1																						
6	1	2	4	4	2	1																					
7	1	2	4	5	4	2	1																				
8	1	2	4	6	6	4	2	1																			
9	1	2	4	7	9	7	4	2	1																		
10	1	2	4	7	10	10	7	4	2	1																	
11	1	2	4	7	11	13	11	7	4	2	1																
12	1	2	4	7	12	16	16	12	7	4	2	1															
13	1	2	4	7	12	17	20	17	12	7	4	2	1														
14	1	2	4	7	12	18	24	24	18	12	7	4	2	1													
15	1	2	4	7	12	19	27	31	27	19	12	7	4	2	1												
16	1	2	4	7	12	19	28	35	35	28	19	12	7	4	2	1											
17	1	2	4	7	12	19	29	39	45	39	29	19	12	7	4	2	1										
18	1	2	4	7	12	19	30	42	53	53	42	30	19	12	7	4	2	1									
19	1	2	4	7	12	19	30	43	57	64	57	43	30	19	12	7	4	2	1								
20	1	2	4	7	12	19	30	44	61	75	75	61	44	30	19	12	7	4	2	1							
21	1	2	4	7	12	19	30	45	64	83	94	83	64	45	30	19	12	7	4	2	1						
22	1	2	4	7	12	19	30	45	65	87	106	106	87	65	45	30	19	12	7	4	2	1					
23	1	2	4	7	12	19	30	45	66	91	117	131	117	91	66	45	30	19	12	7	4	2	1				
24	1	2	4	7	12	19	30	45	67	94	125	150	150	125	94	67	45	30	19	12	7	4	2	1			
25	1	2	4	7	12	19	30	45	67	95	129	162	181	162	129	95	67	45	30	19	12	7	4	2	1		
26	1	2	4	7	12	19	30	45	67	96	133	173	208	208	173	133	96	67	45	30	19	12	7	4	2	1	
27	1	2	4	7	12	19	30	45	67	97	136	181	227	253	227	181	136	97	67	45	30	19	12	7	4	2	1

Table 4.1 The rank of $Q_{g,i}$ for $2 \leq g \leq 27$ and $0 \leq i \leq 26$.

l	0	1	2	3	4	5	6	7	8	9	10	11
$a(l)$	1	1	2	3	5	6	10	13	18	24	33	41

Table 4.2 The a-function for $0 \leq l \leq 11$. The values for $l \leq 9$ can be found in [10] while $a(10)$ and $a(11)$ are found in [25].

l	0	1	2	3	4	5	6	7	8	9	10	11
n	1	2	3	6	10	14	22	33	45	64	90 (91)	119 (120)
#	8	7	6	6	5	4	4	3	2	2	2 (1)	1 (1)

Table 4.3 n for $0 \leq l \leq 11$. # is the number of g for which n has been computed. The numbers in parentheses are values for which the expected behaviour fails along with how many times that happened for each l.

Suppose that we want to produce relations in $R^i(\mathcal{C}_g)$ by multiplying $c_j(\mathbb{F}_{2g-1} - \mathbb{E}) \in R^j(\mathcal{C}_g^{2g-1})$ by a monomial M and then pushing down. Since the degree drops by $2g-2$ and $c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ has degree j, the degree d of the monomial must be $d = i + 2g - 2 - j$. Choose $q = 2g + 2i - 2j + 1$ and define monomials in the following way.

(a) Define $M_0 = D_{1,2}D_{1,3} \cdots D_{1,q}D_{q+1,q+2}D_{q+3,q+4} \cdots D_{2g-2,2g-1}$,
(b) for $r = 0, 1, \ldots, q-3$, replace $D_{1,q-r}$ by $D_{q-r,q-r+1}$ in M_r to obtain M_{r+1}.

Each M_r is a monomial of degree $i + 2g - 2 - j$ and $M_r c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ will thus give a relation in $R^i(\mathcal{C}_g)$ when pushed down.

The second step is to calculate $M \cdot c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ for suitable choices of j. As stated earlier, we have

$$c(\mathbb{F}_{2g-1}) = (1 + K_1)(1 + K_2 - \Delta_2)(1 + K_3 - \Delta_3) \cdots (1 + K_{2g-1} - \Delta_{2g-1}),$$

and

$$c(\mathbb{E})^{-1} = \sum_{i=0}^{g} (-1)^i \lambda_i.$$

Hence

$$\begin{aligned} c(\mathbb{F}_{2g-1} - \mathbb{E}) = c_0(\mathbb{F}_{2g-1}) + c_1(\mathbb{F}_{2g-1}) - \lambda_1 c_0(\mathbb{F}_{2g-1}) + c_2(\mathbb{F}_{2g-1}) - \\ - \lambda_1 c_1(\mathbb{F}_{2g-1}) + \lambda_2 c_0(\mathbb{F}_{2g-1}) + \cdots \end{aligned}$$

If we identify the degree k part we obtain the formula

$$c_k(\mathbb{F}_{2g-1} - \mathbb{E}) = \sum_{i=0}^{k} (-1)^i \lambda_i c_{k-i}(\mathbb{F}_{2g-1}). \tag{1}$$

The following is pointed out in [10]:

$$c_k(\mathbb{F}_n) = c_k(\mathbb{F}_{n-1}) + (K_n - \Delta_n) c_{k-1}(\mathbb{F}_{n-1}).$$

No term of $c_j(\mathbb{F}_{n-1})$ has a factor K_n or $D_{i,n}$. Hence, if P is a polynomial in K_i and $D_{i,j}$ then, $\pi_{n,n*}(P \cdot c_j(\mathbb{F}_{n-1})) = \pi_{n,n*}(P) \cdot c_j(\mathbb{F}_{n-1})$. Putting these pieces together yields the following formula:

$$\pi_{n,n*}(M c_k(\mathbb{F}_n)) = \pi_{n,n*}(M) c_k(\mathbb{F}_{n-1}) + \pi_{n,n*}(M(K_n - \Delta_n)) c_{k-1}(\mathbb{F}_{n-1}). \tag{2}$$

Using formulas (1) and (2), the computations become more manageable.

Finally, one may use Lemmas 3.1 and 3.2 along with formulas (1) and (2) to push the relations down to $R(\mathcal{C}_g)$.

Several Maple procedures has been written for performing these computations. These procedures has then been used to find the necessary number of relations for $2 \leq g \leq 9$. No higher genera have been attempted since the computations are expected to take unfeasibly long time. Below, we present the relations for $g = 2$, 3 and 4. The other relations, as well as the Maple code, are available from the author upon request.

$g = 2$

Since $\kappa_0 = 2g - 2 = 2$, there should be no relation in degree zero. In degree one there should be one relation. Multiplying $c_2(\mathbb{F}_3 - \mathbb{E})$ by $D_{2,3}$ and pushing down to $R^*(\mathcal{C}_2)$ yields the relation $\frac{5}{3}\kappa_1 = 0$. Hence, $K \neq 0$ and $\kappa_1 = 0$. This is no surprise, since κ_1 is the pullback of κ_1 in $R^*\mathcal{M}_g$, which is zero by [10]. The result also follows from Theorem 3.3 and Theorem 3.4.

$g = 3$

Since $g/3 = 1$ we should have no relations in degrees zero and one. In degree two we should have three relations (and will have, by Theorems 3.3 and 3.4). Multiplying $c_3(\mathbb{F}_5 - \mathbb{E})$ with $D_{1,2}D_{1,3}D_{4,5}$ respectively $D_{1,2}D_{3,4}D_{4,5}$ and pushing down to $R^*(\mathcal{C}_3)$ yields the relations

$$42K^2 - \frac{21}{2}K\kappa_1 + \frac{7}{48}\kappa_1^2 = 0, \quad 126K^2 - \frac{63}{2}K\kappa_1 + \frac{41}{48}\kappa_1^2 - 6\kappa_2 = 0.$$

Multiplying $c_4(\mathbb{F}_5 - \mathbb{E})$ with $D_{2,3}D_{4,5}$ and pushing down yields the relation

$$56K^2 - 14K\kappa_1 + \frac{47}{12}\kappa_1^2 - 20\kappa_2 = 0.$$

These three relations are linearly independent, so we are done. If we solve the equations we see that

$$\kappa_1^2 = \kappa_2 = 0, \quad \text{and} \quad K\kappa_1 = 4K^2.$$

$g = 4$

We expect to find two relations in degree 2 and six in degree 3. Multiplying $c_4(\mathbb{F}_7 - \mathbb{E})$ with $D_{1,2}D_{1,3}D_{4,5}D_{6,7}$ respectively $D_{1,2}D_{3,4}D_{4,5}D_{6,7}$ and pushing down yields the relations

$$420K^2 - 70K\kappa_1 + \frac{115}{6}\kappa_1^2 - 150\kappa_2 = 0, \quad 120K^2 - 20K\kappa_1 + \frac{10}{3}\kappa_1^2 - 20\kappa_2 = 0.$$

These relations are linearly independent so we are done in degree 2. We solve the equations to obtain

$$\kappa_1^2 = \frac{32}{3}\kappa_2, \quad \text{and} \quad K\kappa_1 = 6K^2 + \frac{7}{9}\kappa_2.$$

Note that the first of these relations is the relation we obtained in $R(\mathcal{M}_4^2)$ in Example 4.1.

In degree 3 we have the six linearly independent relations which can be written as

$$\kappa_3 = \kappa_2\kappa_1 = \kappa_1^3 = 0, \quad K_1^2\kappa_1 = \frac{32}{3}K_1^3, \quad K_1\kappa_1^2 = 64K_1^3, \quad K_1\kappa_1 = 6K_1^3.$$

4.5 Concluding Remarks

We have already mentioned that the method for finding relations is too inefficient to generate relations for genus 10 and higher, at least in its present implementation. It is therefore desirable to find a way to overcome this obstacle.

One possible way to do this would be to encode the relations in a generating function and then retrieve coefficients. This approach has been applied quite extensively in the case of $R(\mathcal{M}_g)$, for instance by Ionel, [23], and more recently by Pandharipande and Pixton, [31]. It therefore seems very plausible that similar methods may be applied in the case of $R(\mathcal{C}_g)$.

Some attempts in this direction have already been made. One may note that Ionel derives some of her results in [23] from a relation in $R(\mathcal{C}_g)$. However, this just gives one relation in every other codimension so it does not help us very much, at least not without further analysis.

Perhaps more interesting is the fact that Theorem 4 in the article [31] of Pandharipande and Pixton is derived from the results of Ionel. It is therefore not unreasonable to expect that one might find a corresponding generating function for relations in $R(\mathcal{C}_g)$. This is however not so easy, since one has to guess several families of coefficients, but still not completely hopeless, since these families have to satisfy several conditions.

Another possible direction is to note the close relationship between the functions $a(l)$ and $b_G(l)$ described earlier in this chapter since $a(l)$ counts the number of relations in $R^k(\mathcal{M}_g)$ for certain g and k and $b_G(l)$ is supposed to count the number of relations in $R^k(\mathcal{C}_g)$ for certain g and k. Recall that

$$b(l) = a(l) + a(l-1) + a(l-3) + a(l-4) + \cdots .$$

We interpret the term $a(l)$ as corresponding to the relations on $R^k(\mathcal{C}_g)$ that are pulled back from $R^k(\mathcal{M}_g)$. With this interpretation, is not far-fetched to expect that the other terms might correspond to classes of relations on $R(\mathcal{M}_g)$ in a similar way.

Finally, it would of course be interesting to investigate $R(\mathcal{C}_{25})$ in greater detail. Perhaps one could then explain the discrepancy between the actual (expected) number of relations and $b_G(l)$ at $l = 10$.

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