Löwenheim-Skolem theorems and Choice principles

Christian Espíndola*

It has been known since 1920 ([6], [2]) that the the axiom of choice played a crucial rôle in the proof of the following form of Löwenheim-Skolem theorem:

Theorem 1. $\mathbf{LS}(\aleph_\mathbf{0})$*: Every model $\mathcal{M}$ of a first order theory T with countable signature has an elementary submodel $\mathcal{N}$ which is at most countable.*

As noted by Skolem himself, the condition that the countable model for T be an elementary submodel of $\mathcal{M}$ made an apparently essential use of the axiom of choice, by relying on the so called *Skolem functions*, which can be found in the modern standard proofs of the theorem. A careful examination of the proof shows that the axiom of dependent choice is sufficient already (see [3]) and in fact it happens to be equivalent to it, as shown in the following[1]:

Theorem 2. $\mathbf{LS}(\aleph_0)$ *is equivalent to the Axiom of Dependent Choice.*

Proof. Let S be a nonempty set with a binary relation R such that for every $x \in S$, the set $\{y \in S \mid xRy\}$ is nonempty. Consider the theory T over the language $\mathcal{L} = \{R\}$ that contains a binary relation symbol, and whose only non logical axiom is $\forall x \exists y R(x, y)$; then S is a model for T with the obvious interpretation of the relation symbol. Since T has a countable signature, by $\mathbf{LS}(\aleph_\mathbf{0})$ it contains a countable submodel $\mathcal{N}$, and hence, there is a bijection between the underlying set N of $\mathcal{N}$ and either the set of natural numbers or some finite ordinal. If f is such a bijection, we can now recursively define a sequence by putting $x_0 = f^{-1}(\min\{f(n) \mid n \in N\})$ and $x_n = f^{-1}(\min\{f(n) \mid n \in N \wedge x_{n-1}Rn\})$. $\square$

$\mathbf{LS}(\aleph_\mathbf{0})$ can be generalized to theories over signatures that have arbitrary cardinalities. This leads us to the following:

Theorem 3. *Let $\mathbf{LS}(\kappa)$ be the statement: "Every model $\mathcal{M}$ of a first order theory T whose signature has cardinality κ has an elementary submodel $\mathcal{N}$ whose universe has cardinality less or equal than κ". We have:*

1. *The assertion: "$\mathbf{LS}(\kappa)$ holds for every infinite cardinal κ" is equivalent, over ZF, to the Axiom of Choice (AC).*

2. *The assertion: "$\mathbf{LS}(\kappa)$ holds for every infinite aleph κ" is equivalent, over ZF, to the Axiom of Choice for well orderable families of nonempty sets (AC_{WO}).*

Proof. 1. It suffices to prove that the assertion implies AC. By (1) in the theorem of [7], it is enough to prove that if a sentence ϕ has a model $\mathcal{M}$ of cardinality κ, it also has a model of cardinality μ for every $\aleph_0 \leq \mu \leq \kappa$. Consider the language that contains symbols from ϕ as well as μ many constant symbols c_i, and the theory T consisting of the sentence ϕ together

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[1] A referee indicated the author that the proof exposed here had already been published in [1], excercice 13.3. This result however is not widely known, as it was missed in the monography [3]

with the sentences $c_i \neq c_j$ for $i \neq j$. Since $\aleph_0 \leq \mu$ and there are only a finite number of symbols in ϕ, the new signature has cardinality μ. Also, since $\mu \leq \kappa$, $\mathcal{M}$ contains μ many different elements and hence can be turned into a model of T by using these as the interpretations of the constant symbols. By the assertion, $\mathcal{M}$ has an elementary submodel of T of cardinality at most μ, and hence of cardinality exactly μ (since it must contain at least μ different elements). Since this submodel is also a model of ϕ, the proof is complete.

2. To prove that AC_{WO} implies the assertion, note that the usual proof of Löwenheim-Skolem theorem can be adapted to prove $\mathbf{LS}(\kappa)$, provided that κ is well-orderable and that we have the Axiom of Choice for families of κ sets (AC_κ) and the Axiom of Dependent Choice (DC). But these requirements are fulfilled since AC_{WO} implies DC (see theorem 8.2, pp. 121 of [4]). Let us now prove the converse, i.e., that the assertion implies AC_{WO}. Let κ be any aleph, and consider a set of κ unary relation symbols R_i and the theory consisting of the sentences $\exists x R_i(x)$ for each i. Given a family of κ nonempty sets U_i, their union $\cup_{i \in \kappa} U_i$ can be turned into a model for that theory under the interpretation $R_i(x) \iff x \in U_i$. By the assertion, there is an elementary submodel whose cardinality is at most κ (in particular, it is well-orderable). Hence, a choice function for he family is given by considering, for each i, the least element x in the submodel that satisfies $R_i(x)$.
$\square$

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