Carlson's proof of Marden's theorem*

The following is a short and elementary proof of Marden's theorem as explained in the solution of an exercise in Fritz Carlson's book "Geometri" (in Swedish, 1943). It uses the same background assumed in Dan Kalman AMM paper "An elementary proof of Marden's theorem", but is much simpler and direct than Kalman's proof (who seems to have been unaware of Carlson's proof). Since the book only exists in Swedish, I copy here the argument:

Theorem 1. *If $P(z)$ is a polynomial of degree 3 whose roots are three non-collinear points in the complex plane, vertices of a triangle, the roots of $P'(z)$ are the two foci of the (unique) inscribed ellipse tangent to the sides of the triangle at their midpoints.*

Proof. Any change of cartesian coordinates $z \mapsto z'$ can be described by a rotation $z \mapsto \epsilon z$ for $|\epsilon| = 1$, followed by a translation $z \mapsto z + e$. In the new coordinates, the polynomial becomes $P(z') = P(e + \epsilon z) = Q(z)$, and since $dQ/dz = \epsilon dP/dz'$, the roots of the derivatives of P and Q coincide, so we can choose our coordinate system as we wish without loss of generality. Given an arbitrary triangle with its inscribed ellipse of semi-axes a and b, choose the coordinates origin O in the centre of the ellipse and the two axes Ox and Oy along the ellipses axes, so that the two foci have coordinates $(\pm c, 0)$, with $c = \sqrt{a^2 - b^2}$. Let $z_k = x_k + iy_k$ be the vertices of the triangle. A linear transformation stretching the axis Oy by b/a maps the ellipse into a circle and the vertices to new points $\zeta_k = \xi_k + i\eta_k$, such that $x_k = \xi_k$ and $y_k = (b/a)\eta_k$. Since the midpoints of the sides of the original triangle are mapped into the midpoints of the sides of the new triangle, which has an inscribed circle, this latter triangle is equilateral, so that $\Sigma\zeta_i$ (and hence Σz_i) is zero. This means that for the original triangle, $P(z) = z^3 + pz + q$. Also, the ζ_k must hence be the roots of an equation $\zeta^3 = C$, from which we get that $\Sigma\zeta_i\zeta_{i+1} = 0$ and $\Sigma\zeta_i^2 = 0$. This latter implies that $\Sigma\xi_i^2 = \Sigma\eta_i^2$ and $\Sigma\xi_i\eta_i = 0$. Also, since a is the radius of the inscribed circle in the equilateral triangle, we have $\xi_k^2 + \eta_k^2 = (2a)^2$, from which we get $\Sigma\xi_i^2 = \Sigma\eta_i^2 = (1/2)3(2a)^2 = 6a^2$, and, consequently, $\Sigma x_i^2 = (a^2/b^2)\Sigma y_i^2 = 6a^2$. Finally, $2p = 2\Sigma z_i z_{i+1} = (\Sigma z_i)^2 - \Sigma z_i^2 = -\Sigma z_i^2 = -\Sigma x_i^2 + \Sigma y_i^2 - 2i\Sigma x_i y_i = -6a^2 + 6b^2 = -6c^2$. Therefore, $P'(z)$ vanishes for $3z^2 = -p = 3c^2$, i.e., for $z = \pm c$, QED. □

References

[1] Carlson, Fritz: *Lärobok i geometri, del 1-2* - Gleerup, Lund (1943)

[2] Kalman, D.: *An elementary proof of Marden's theorem* - American Mathematical Monthly, vol. 115, pp. 330-338 (2008)

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