Constructive completeness and non-discrete languages

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Abstract

We give an analysis and generalizations of some long-established constructive completeness results in terms of categorical logic and presheaf and sheaf semantics. The purpose is in no small part conceptual and organizational: from a few basic ingredients arises a more unified picture connecting constructive completeness with respect to Tarski semantics, to the extent that it is available, with various completeness theorems in terms of presheaf and sheaf semantics (and thus with Kripke and Beth semantics). From this picture are obtained both ("reverse mathematical") equivalence results and new constructive completeness theorems; in particular, the basic set-up is flexible enough to obtain strong constructive completeness results for languages of arbitrary size and languages for which equality between the elements of the signature is not decidable.

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1 Introduction

Starting with the Gödel-Kreisel theorem, it has long been well known that the classically "standard" semantics—Tarski structures for classical first-order logic (FOL), and Kripke or Beth structures for intuitionistic FOL—are insufficient in a constructive metatheory. For instance, the assumption of strong completeness for intuitionistic FOL with respect to Tarski, Kripke, or Beth semantics in a metatheory such as **IZF**, **HAS**, or **HAA**[1] implies the law of excluded middle (LEM), while weak completeness implies Markov's principle (MP) (see [19], [12], [20]). On the other hand, constructive completeness theorems exist e.g. with respect to formal space valued models, and in sheaf toposes more generally (see e.g. [21], [6], [11]). In a sense intermediate between sheaf semantics and the standard semantics of Tarski and Kripke, completeness was also shown (albeit assuming the Fan Theorem) to

[1]Intuitionistic Zermelo Fraenkel set theory; full intuitionistic second-order arithmetic; intuitionistic second-order arithmetic with arithmetic comprehension

hold for "countable" intuitionistic first-order theories with respect to fallible Kripke and Beth semantics by Veldman [27] and by de Swart [8].

The insufficiency of the standard semantics can be taken to suggest that constructive model theory should instead be carried out with respect to sheaf models, or some variant thereof. On the other hand, the theorems of Veldman and de Swart indicate that ordinary Tarski semantics can have an important role to play. Thus the underlying conceptual and motivating question of this paper is the role of Tarski semantics in constructive model theory, or how much "mileage" one can get out of ordinary Tarski-models in a constructive setting.

Developments of logic in a constructive setting usually assume that the signatures are, if not in some sense countable, then at least discrete; i.e. that decidability (LEM) holds for equality of the basic function and relation symbols. This precludes various classical constructions, such as adding the elements of an arbitrary domain as constants to the language. Or similarly in categorical logic, constructing the internal language of an arbitrary category. Here, we drop this restriction.

In a classical metatheory (with choice), the link between Tarski completeness for classical first-order logic (or its so-called coherent fragment) and Kripke completeness for intuitionistic first-order logic is put to light in a theorem attributed to A. Joyal in [18]. The theorem involves, among other things, the technique of considering a theory in terms of an "approximation" in a weaker fragment, known sometimes, or in some cases, as *Morleyization*. However, the theorem is not immediately applicable in a constructive setting as it relies on the assumption of Tarski completeness for the coherent fragment of FOL. The proof ([18, Thm. 6.3.5]) also uses classical techniques. Nevertheless, while Tarski completeness for the coherent fragment fails to hold in a constructive meta-theory, constructive completeness results for less expressive fragments of FOL exist. By giving a constructive formulation and proof of Joyal's theorem, and using the same "approximation" technique, these can be exploited to give completeness results for stronger fragments of FOL, and for FOL itself, in suitable pre-sheaf and sheaf toposes. For instance, the aforementioned completeness theorems for fallible Kripke and Beth semantics can be recovered (and given new proofs) in this way. The purpose and aim of this paper is in no small part to "tell this story"; that is, to give a unified and conceptual account connecting constructive Tarski completeness, to the extent that it exists, with completeness results in terms of traditionally studied Kripke and Beth-style models. And, furthermore, to give this account using essentially only the two basic ingredients of Joyal's theorem and of Morleyization to fragments for which Tarski completeness holds. In addition to displaying the connections between already established results, such a clearer conceptual picture can also serve to suggest further, and new, ones. As an instance of this, we extend constructive completeness results to languages and theories that are not enumerable, and for which equality of non-logical symbols in the language can not be decided. (We also do not, in general, assume that the sentence $\exists x.\, x = x$ is valid). In particular, we give a strong completeness theorem for the disjunctive-free fragment of FOL, over such languages, with respect to fallible Kripke models, and a strong completeness theorem for full FOL with respect to fallible, "generalized" Beth models. We also draw attention to a conceptual link between Beth semantics and a "least coverage forcing the correct interpretation of disjunctions", and give a constructive version and proof of the completeness theorem of [10] with respect to Beth models in which only the forcing clause for disjunction is used.

The paper is structured into the following parts. Section 2 contains preliminaries and notes on notation and terminology, and recapitulates Tarski completeness for the regular fragment and enumerable coherent theories. The regular completeness theorem for arbitrary theories and signatures is more fully presented in [9]. The completeness theorem for enumerable positive coherent theories is known, but is included for conceptual self-containement. (We also note the equivalence between this theorem and the Fan theorem, and draw a corollary concerning completeness for classical first-order theories.) Section 3 gives a constructive reformulation and proof of Joyal's theorem. Section 4 introduces and analyses certain coverages on the category of models, and gives a covering lemma which allows Joyal's theorem to be stated with respect to a poset of structures and homomorphic inclusions. This then linked with the instances of Tarski completeness in Section 2 to give the aforementioned Kripke and Beth completeness results theories over non-discrete languages. The specialization to the constructive version of the theorem of [10] marks the end.

2 Constructive Tarski completeness

2.1 Preliminiaries

2.1.1 Theories, models and diagrams

In what follows we shall fix our metatheory to be **IZF** and simply use "constructive" to mean that we are working in this setting. Correspondingly, "classically" means in the metatheory **ZFC**. We fix the following terminology, conventions, and notation. By "finite" and "countable" we mean cardinal finite and isomorphic to $\mathbb{N}$, respectively. A "list" is a finite list. $\mathrm{Im}(\mathbf{a})$ denotes the set of elements of the list $\mathbf{a}$, and $l(\mathbf{a})$ its length. When deemed safe, we shorten $\mathbf{a} \in A^{l(\mathbf{a})}$ to $\mathbf{a} \in A$. A subset of a set A is decidable if it is given in terms of a function $f : A \twoheadrightarrow 2$ as $\{a \in A \mid f(a) = 1\}$ and semi-decidable if it is given in terms of a function $f : A \twoheadrightarrow 2^{\mathbb{N}}$ as $\{a \in A \mid \exists n.\, (f(a))(n) = 1\}$. By an *enumerable* set we mean a semi-decidable subset of a (perhaps implicit) countable set.

Let Σ be a first-order signature. We generally assume that Σ is *relational* in the sense that it has no function symbols. Thus all functions and constants are taken to be represented by relations and appropriate axioms over Σ. Furthermore, we assume, that Σ is single-sorted. Thus Σ is simply a set of relation symbols with associated (finite) arities. We do not assume, unless otherwise stated, that Σ is *discrete*—i.e. that LEM holds for equality betweeen the elements of Σ. (As usual, however, the logical symbols, including variables, are discrete, and disjoint from Σ). Following [11, D1], we consider theories in FOL and fragments of FOL formulated in terms of sequents of the form $\phi \vdash_{\mathbf{x}} \psi$. These can be read as $\forall \mathbf{x}.\, \phi \to \psi$. The list of variables $\mathbf{x}$ is required to be a *context* for both ϕ and ψ in the sense that it is a list of distinct variables containing (at least) the free variables of the formula (see [11, D1.1.4]). We write $\mathbf{x}.\phi$ for a *formula-in-context.* We write $[\mathbf{x} \mid \phi]$ for a formula-in-context identified up to α-equivalence. (That is, up to renaming of bound variables and variables in the context, as in [11, D1.4].) The context, in both cases, is *canonical* if it contains only the free variables of the formula, listed in order of first-appearance. The logic is "free", in the sense that the sequent $\top \vdash \exists x.\, x = x$ is, in general, not derivable. The main fragments we shall be referring to are: the *Horn* fragment[2], consisting of sequents with formulas over Σ involving only the logical constants $\top$ and $\wedge$; the *regular* fragment $\top, \wedge, \exists$; the *regular*$_\perp$ fragment $\top, \wedge, \exists$, and $\perp$; the *positive coherent* fragment $\top, \wedge, \vee$, and $\exists$; the *coherent* fragment $\top, \wedge, \vee, \exists$, and $\perp$; and, of course, full FOL. Deduction rules and further details can be found in [11, D1.3]. The distinguished relation symbol $=$ of equality is included in all languages under consideration. If a theory $\mathbb{T}$ proves a sequent $\phi \vdash_{\mathbf{x}} \psi$ we sometimes write $\phi \vdash^{\mathbb{T}}_{\mathbf{x}} \psi$ instead of $\mathbb{T} \vdash (\phi \vdash_{\mathbf{x}} \psi)$. Provable in the empty theory is then written $\phi \vdash^{\emptyset}_{\mathbf{x}} \psi$.

One would usually say that a theory is regular, for instance, if it is axiomatizable by regular sequents. Thus a Horn theory would also be a regular theory etc. For brevity, however, we also mean to indicate what fragment we are considering when we say that a theory is this or that. Thus when we say e.g. that $\mathbb{T}$ is a coherent theory and ϕ is a formula of $\mathbb{T}$, we mean, in particular, that ϕ a formula in the coherent fragment over the signature of $\mathbb{T}$. In the same vein, if we say that a theory is discrete, we mean that it is over a discrete signature. If we say that a theory is enumerable we mean both that it is over a enumerable signature and that the set of axioms is enumerable

We say that a coherent sequent is on *canonical form* if it is on the form $\phi \vdash_{\mathbf{x}} \exists \mathbf{y}_0 \psi_0 \vee \ldots \vee \exists \mathbf{y}_n \psi_n$ where ϕ and all ψ_i are Horn formulas, or on the form $\phi \vdash_{\mathbf{x}} \perp$ where ϕ is Horn. A regular sequent is on canonical form if it

[2] To prevent confusion with otherwise standard usage of "Horn clause" and "Horn formula", note the usage here (following [11]) of "Horn formula" as simply a formula which is a conjunction of atomic formulas, and "Horn sequent" as simply a sequent with such Horn formulas as antecedent and consequent.

is so as a coherent sequent. Every coherent (regular$_\bot$, regular) theory can be axiomatized by coherent (regular$_\bot$, regular) sequents on canonical form (see *ibid.*), and we assume that they are.

By *Tarski structure* for a signature, and *Tarski model* for a theory, we mean the usual notion of a domain set with interpretations of the relation symbols in terms of subsets, and the interpretations of the connectives $\bot$, $\top$, $\wedge$, $\vee$, and $\exists$ by the usual set-theoretic interpretations (as well as $\rightarrow$ and $\forall$, but we do not actually consider Tarski models for anything above the coherent fragment, with the exception of Corollary 2.2.11). The domain need not be inhabited or non-empty. As for the interpretation of the equality relation, we reserve "structure" and "model" for the case where equality is interpreted as the identity relation, and use *diagram* and *model diagram* for the case where equality is interpreted as a congruence relation[3]. That is to say, a diagram for a relational signature Σ consists of a Σ-structure $\mathbf{M}$ together with an equivalence relation E on $|\mathbf{M}|$ which respects the relations interpreting the symbols of Σ. The interpretation $[\![\mathbf{x} \mid \phi]\!]^{\mathbf{M}}$ of a formula-in-context $\mathbf{x}.\phi$ is then defined in the usual way, but interpreting $=$ as E.

In part to notationally distinguish diagrams, in this sense, from structures, we consider and write a diagram $\mathbf{M}$ as a pair (D, F) where $D = |\mathbf{M}|$ is the domain and $F = \left\{\langle[\mathbf{x} \mid \phi], \mathbf{d}\rangle \;\middle|\; \phi \text{ is Horn, and } \mathbf{d} \in [\![\mathbf{x} \mid \phi]\!]^{\mathbf{M}}\right\}$, where $[\![\mathbf{x} \mid \phi]\!]^{\mathbf{M}}$ is the extension of $\mathbf{x}.\phi$ in $\mathbf{M}$. We refer to an element of F as a *fact*. For a signature Σ and theory $\mathbb{T}$ we write $\mathrm{Str}(\Sigma)$ and $\mathrm{Mod}(\mathbb{T})$ for the category of structures and homomorphisms and models and homomorphisms, respectively. We write $\mathrm{Diag}(\Sigma)$ and $\mathrm{MDiag}(\mathbb{T})$ for the category of diagrams and homomorphisms and model diagrams and homomorphisms, respectively, where a homomorphism $h : (D_1, F_1) \longrightarrow (D_2, F_2)$ between diagrams is a left-total relation $h \subseteq D_1 \times D_2$ such that

1. $h(d_1, d_2) \wedge \langle[x, y \mid x = y], d_2, d_2'\rangle \in F_2 \rightarrow h(d_1, d_2')$; and
2. $\langle[\mathbf{x} \mid \phi], \mathbf{d}_1\rangle \in F_1) \wedge h(\mathbf{d}_1, \mathbf{d}_2) \rightarrow (\langle[\mathbf{x} \mid \phi], \mathbf{d}_2\rangle \in F_2)$, for all (atomic) Horn formulas ϕ over Σ and all $\mathbf{d}_1 \in D_1$ and $\mathbf{d}_2 \in D_2$.

(Here $h(\mathbf{d}_1, \mathbf{d}_2)$ stands for the expected conjunction. As further notational shortcuts, we allow ourselves to use function notation for homomorphisms between diagrams when no confusion threatens. For instance we might write $\phi[h(d_1)/x]$ instead of $\forall d_2.\, h(d_1, d_2) \rightarrow \phi[d_2/x]$. We sometimes write $\phi[\mathbf{d}/\mathbf{x}] \in F$, or simply $\phi[\mathbf{d}] \in F$, instead of $\langle\mathbf{d}, [\mathbf{x} \mid \phi]\rangle \in F$, for brevity.) Then there is an adjoint equivalence

$$\begin{tikzcd} \mathrm{Str}(\Sigma) \arrow[r, bend left, "i"] \arrow[r, phantom, "\simeq"] & \mathrm{Diag}(\Sigma) \arrow[l, bend left, "q"] \end{tikzcd}$$

[3] We avoid using "diagram" when extending a language with a structure, talking instead of the language and theory of the structure.

where i is the inclusion and q is by taking quotients, in the expected way. The unit $\mathrm{k} : (D, F) \twoheadrightarrow q(D, F)$ preserves and reflects the intepretation of formulas and the truth of sequents. Thus, in particular, the equivalence restricts to models, $\mathrm{Mod}(\mathbb{T}) \simeq \mathrm{MDiag}(\mathbb{T})$.

The diagram notation may seem cumbersome at first, but it is convenient for working with presentations of structures. Let a *presentation* of a diagram, or *pre-diagram*, be a pair (D, F) where D is a set and F is set of facts over Σ and D, in the sense above; that is to say, F is a set of pairs $\langle [\mathbf{x} \mid \phi], \mathbf{d} \rangle$ with ϕ Horn and $\mathbf{d} \in D^{l(\mathbf{x})}$. For a pre-diagram (D, F) the least diagram containing it is the diagram *generated by* (D, F). A homomorphism of pre-diagrams is a left-total relation that is a homomorphism of the generated diagrams. The satisfaction relation $\vDash$ for pre-diagrams is defined as satisfaction in the generated diagram.

For (D, F) is a pre-diagram, the theory $\mathbb{D}_{(D,F)}$ of (D, F) is defined as expected by extending Σ with D as constants (or unary predicates, see below) and letting $\{\top \vdash \phi[\mathbf{d}/\mathbf{x}] \mid \langle [\mathbf{x} \mid \phi], \mathbf{d} \rangle \in F\}$ be axioms. The diagram generated by (D, F) can then be defined as $(D, \{\langle [\mathbf{x} \mid \phi], \mathbf{d} \rangle \mid \top \vdash^{\mathbb{D}_{(D,F)}} \phi[\mathbf{d}/\mathbf{x}]\})$. If $\mathbb{T}$ is a theory over Σ we write $\mathbb{T}_{(D,F)}$ for the union of $\mathbb{T}$ and $\mathbb{D}_{(D,F)}$.

When constants are not discrete, replacing constants with variables in proofs becomes more problematic; that is, one cannot in general replace the same constant with the same variable throughout a formula. This makes the interplay between a theory and the theory of one of its diagrams a little more intricate. The following lemmas, which are completely straightforward for discrete signatures, are shown also to hold for non-discrete signatures in [9] and stated here for reference.

Lemma 2.1.2 *Let $\mathbb{T}$ be a theory over Σ. Let C be a set of constants disjoint from Σ, and write $\Sigma^C = \Sigma \cup C$. Suppose $\phi \vdash_{\mathbf{x}} \psi$ is a first-order sequent over Σ^C which is provable from axioms in $\mathbb{T}$. Then there exists a sequent $\phi' \vdash_{\mathbf{x},\mathbf{y}} \psi'$ over Σ and a "valuation" function $\mathrm{f} : \mathbf{y} \twoheadrightarrow C$ such that;*

(i) $\phi' \vdash_{\mathbf{x},\mathbf{y}} \psi'$ is provable from (the same) axioms in $\mathbb{T}$; and

(ii) $\phi'[f] = \phi$, and $\psi'[f] = \psi$.

Lemma 2.1.3 *Let $\mathbb{T}$ be a theory over Σ and (D, F) be a (pre-)diagram. Let $\mathbf{x}.\psi$ and $\mathbf{x}, \mathbf{y}.\phi$ be first-order Σ-formulas-in-context. Let $\mathbf{c}$ be a tuple of elements in D. Suppose $\mathbb{T}_{D,F}$ proves the sequent $\phi[\mathbf{c}/\mathbf{y}] \vdash_{\mathbf{x}} \psi$. Then there is a regular formula ξ in context $\mathbf{y}$ in Σ such that $(D, F) \vDash \xi[\mathbf{c}/\mathbf{x}]$ and $\mathbb{T}$ proves the sequent $\xi \wedge \phi \vdash_{\mathbf{x},\mathbf{y}} \psi$.*

As mentioned in the beginning, we generally assume that signatures are relational and thus that any language with function symbols has been translated into an equivalent one without them (cf. [3]). This is in principle so also when extending a signature with the language of one of its diagrams, as above. In practice this becomes burdensome, however, and we leave the translation implicit and swept under the rug.

Finally, we say that a diagram (D, F) is a *subdiagram* of (D', F') if $D \subseteq D'$ and $F \subseteq F'$. We write $(D, F) \subseteq (D', F')$. Note that the inclusion $D \subseteq D'$ induces a homomorphism $\mathrm{i} : (D, F) \longrightarrow (D', F')$ by $i(d, d') \Leftrightarrow (d = d') \in F'$. (This homomorphism need not be a monomorphism.) A diagram is *finite* if the domain is finite and the interpretations of $=$ and all relation symbols are finite. For enumerable signature Σ, a diagram is *enumerable* if the domain is enumerable and the interpretations of $=$ and all relation symbols are enumerable. For discrete signature Σ, a diagram is *discrete* if the domain is discrete. We define a *bounded diagram* to be a triple (D, F, n) where (D, F) is a diagram, $n \in \mathbb{N}$, and the elements of the domain D are pairs where the second component is a natural number less than or equal to n. Clearly, any diagram is canonically isomorphic to a bounded one (with bound 0, say). We shall in fact mostly restrict to bounded diagrams, but leave the bound n notationally implicit. Let $\mathrm{Diag}_b(\Sigma)$ be the category of bounded diagrams and diagram homomorphisms.

2.1.4 Syntactic categories, Morleyization, and exploding models

Recall from e.g. [11, D1.4] the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of a theory $\mathbb{T}$, consisting of formulas-in-context of the language of the theory. For coherent $\mathbb{T}$, the category $\mathcal{C}_{\mathbb{T}}$ is a coherent category and models of $\mathbb{T}$ in a coherent category $\mathcal{D}$ can be considered as coherent functors $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{D}$. Similarly for e.g. regular and first-order theories, see *loc. cit.* for precise statements and details. The functor $\mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{D}$ is *conservative* (see e.g. [11]) if and only if the corresponding model is conservative, or complete; that is, if only provable sequents are true.

We refer to the rewriting of a theory into a theory of a less expressive fragment as "Morleyizing" the theory, after the rewriting of a classical first-order theory as an equivalent coherent theory (as in e.g. [11, D1.5.13]). In categorical terms, the syntactic category $\mathcal{C}_{\mathbb{T}}$ of, say, an intuitionistic first-order theory $\mathbb{T}$ is a Heyting category, and thus also a coherent and regular category. The Heyting category $\mathcal{C}_{\mathbb{T}}$ therefore has an internal coherent theory $\mathbb{T}^{coh}_{\mathcal{C}_{\mathbb{T}}}$ and an internal regular one $\mathbb{T}^{reg}_{\mathcal{C}_{\mathbb{T}}}$. These theories are equivalent in the sense that their syntactic categories are equivalent.

$$\mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^{coh}_{\mathcal{C}_{\mathbb{T}}}} \simeq \mathcal{C}_{\mathbb{T}^{reg}_{\mathcal{C}_{\mathbb{T}}}}$$

The categories of models of these theories will in general not be the same (unless we are considering classical theories and their coherent Morleyzations, see loc.cit.). Nevertheless, considering the category of models of the Morleyized theory can be fruitful, not least when the more expressive theory has "too few models" (see [17] for another example).

There is some leeway concerning what precisely one takes the internal language and theory of a category to be. In our case we also start from a given theory and not from a given category. We therefore write down explicitly what we shall take the *regular* and *coherent Morleyizations* of a first-order theory $\mathbb{T}$ over a signature Σ to be. Other fragments are similar.

Let Σ^m be the signature extending Σ with, for each first-order formula ϕ over Σ, in canonical context $\mathbf{x}$, say, a relation symbol $\mathbf{P}_\phi$ with arity the length of $\mathbf{x}$. We write P_ϕ for the atomic formula over Σ^m obtained by assigning $\mathbf{x}$ to the arity of $\mathbf{P}_\phi$. Consider the following axioms.

(*Thry*)
For every sequent $\phi \vdash_{\mathbf{x}} \psi$ provable in $\mathbb{T}$, the axiom

$$P_\phi \vdash_{\mathbf{x}} P_\psi$$

(*Atom*)
For every atomic formula ϕ over Σ in canonical context $\mathbf{x}$,

$$P_\phi \dashv\vdash_{\mathbf{x}} \phi$$

(*True*)

$$P_\top \dashv\vdash \top$$

(*Conj*)
For every conjunction $\theta = \phi \wedge \psi$ over Σ in canonical context $\mathbf{x}$

$$P_\theta \dashv\vdash_{\mathbf{x}} P_\phi \wedge P_\psi$$

(*Exist*)
For every existentially quantified formula $\theta = \exists y.\, \phi$ over Σ in canonical context $\mathbf{x}$

$$P_\theta \dashv\vdash_{\mathbf{x}} \exists y.\, P_\phi$$

(*Disj*)
For every disjunction $\theta = \phi \vee \psi$ over Σ in canonical context $\mathbf{x}$

$$P_\theta \dashv\vdash_{\mathbf{x}} P_\phi \vee P_\psi$$

(*False*)

$$P_\perp \dashv\vdash \perp$$

These axioms define the coherent Morleyization of $\mathbb{T}$. The regular Morleyization is obtained by omitting the Disjunction axiom schema and the False axiom. Notice that in, say, the regular Morleyization of a first-order theory, every regular formula is provably equivalent to an atomic formula; and that the sequent $\phi \vdash_{\mathbf{x}} \psi$ is provable in $\mathbb{T}$ if and only if $P_\phi \vdash_{\mathbf{x}} P_\psi$ is provable in $\mathbb{T}^m$. Notice further that if we Morleyize, say, a regular$_\bot$ theory $\mathbb{T}$ to a regular theory, then $\mathbb{T}^m$ will prove $P_\bot \vdash_{\mathbf{x}} \phi$ for all regular $\mathbf{x}.\phi$ over Σ^m. A model diagram (D, F) such that $P_\bot \in F$ will thus have all possible facts in F. The corresponding quotient will consist of a single point of which everything is true (it must be inhabited since $\mathbb{T}^m$ proves $P_\bot \vdash \exists x. \top$). Thus, or in that sense, it is an *exploding* model diagram of $\mathbb{T}$.

Finally we note that the notion of enumerability for theories is not affected by Morleyization. We display this for reference.

Lemma 2.1.5 *If $\mathbb{T}$ is enumerable then so is $\mathbb{T}^m$.*

2.2 Tarski completeness

The dynamical method of [7], the chase algorithm of [1], and similar methods[4] can be seen as simultaneous proof searches and (at least partial) completions of structures to models. In essence, one proceeds by repeatedly applying the axioms of the theory to the structure and adding the result; thus if, for instance, $\phi[a/x]$ is true in the structure and $\phi \vdash_x \exists y. \psi$ is an axiom, one extends the domain with a fresh element b and the interpretation in the least way such that $\psi[a/x, b/y]$ is true. It is in several cases known, or at least folklore, that such methods can be used constructively to obtain completeness results for fragments of FOL. Although the object theories tend to be assumed countable or at least discrete. We summarize in Section 2.2.1 the relevant results from [9] concerning the construction of a functorial "simultaneous chase" to the case of regular theories with no size or discreteness constraints. Section 2.2.8 displays the equivalence between the Fan theorem and completeness for enumerable positive coherent theories with respect to enumerable model diagrams—equivalently of enumerable coherent theories with respect to "possibly exploding" or "fallible" enumerable model diagrams. This equivalence can to a large extent be derived from the literature. In particular, Veldman's proof [27] (which relies on the Fan Theorem) of fallible Kripke completeness for first-order (enumerable) theories implies also the Tarski-completeness of positive coherent (enumerable) theories. Nevertheless, since we are, conceptually, regarding first-order fallible Kripke completeness as flowing from the Tarski completeness of positive coherent theories, we supply a direct proof of the latter using the Fan Theorem. The converse, that this completeness theorem implies the Fan Theorem, is rather immediate, and we include a very short and simple proof. This should be compared with the (equally short) proof in [14] that the contrapositive model existence theorem for decidable (and countable) classical propositional theories is equivalent to the Fan Theorem. (Note

[4] C.f. also [2] and [16])

that the direction completeness $\Rightarrow$ Fan of Proposition 2.2.13 can be carried out in much weaker meta-theories then IZF.

2.2.1 Chase-complete sets of models for regular theories

Let Σ be a single-sorted relational signature. That is, Σ is a arbitrary set of relation symbols (with assigned arities), not assumed to be of a particular size nor discrete.

Definition 2.2.2 Let $\mathbb{T}$ be a theory over Σ, and $\mathcal{S}$ a class of Σ-diagrams. We say that a functor $\mathrm{Ch} : \mathcal{S} \longrightarrow \mathcal{S}$ is a *chase functor* if for all diagrams $(D, F) \in \mathcal{S}$

1. $(D, F) \subseteq \mathrm{Ch}(D, F)$, naturally in (D, F);
2. $\mathrm{Ch}(D, F) \vDash \mathbb{T}$; and
3. for any regular formula $\mathbf{x}.\psi$ and $\mathbf{d} \in D^{l(\mathbf{x})}$ such that $\mathrm{Ch}(D, F) \vDash \psi[\mathbf{d}/\mathbf{x}]$, there exists a regular formula $\mathbf{x}.\phi$ such that $(D, F) \vDash \phi[\mathbf{d}/\mathbf{x}]$ and $\phi \vdash^{\mathbb{T}}_{\mathbf{x}} \psi$.

Proposition 2.2.3 *Let $\mathbb{T}$ be a regular theory. Then there exists a chase functor* $\mathrm{Ch}\colon \mathrm{Diag}_b(\Sigma) \longrightarrow \mathrm{Diag}_b(\Sigma)$.

Proof A straightforward modification of the construction in [9]. (The modification being that, working with (bounded) diagrams rather than structures, we can have the components $c_{(D,F)} : (D, F) \longrightarrow \mathrm{Ch}(D, F)$ of the natural transformation $c : 1_{\mathrm{Diag}_b(\Sigma)} \longrightarrow \mathrm{Ch}$ be inclusions of diagrams, rather than homomorphisms of structures). $\dashv$

In general, we say that a collection of diagrams is *closed under chase* if there is some chase functor Ch under which it is closed. The construction of [9] shows e.g. that if Σ is a discrete signature, then discrete diagrams are closed under chase. In a classical meta-theory the stronger property holds that for a regular theory the category of models is weakly reflective in the category of structures (cf. e.g. [2], [16]). We return to this and the case of enumerable signatures and theories in Section 2.2.8.

For purposes of Section 3 we would, for given signature Σ and regular theory $\mathbb{T}$, like to restrict to a *set* $\mathcal{S}$ of diagrams which is nevertheless "rich enough" for our purposes. Mainly, this involves being closed under chase, but we add some further conditions. Say that a diagram is a *finitary extension* of a diagram (D, F) if it is generated by a pre-diagram of the form $(D \cup \mathrm{Im}(\mathbf{c}), F \cup \langle [\mathbf{x}, \mathbf{y} \mid \phi], \mathbf{d}, \mathbf{c}\rangle)$, where ϕ is a Horn formula, $\mathbf{d}$ a (possibly empty) list of elements of D, and $\mathbf{c}$ is a (possibly empty) list of elements such that $\mathrm{Im}(\mathbf{c})$ is finite and disjoint from D. Say that a collection $\mathcal{S}$ of diagrams is *chase-complete* if: the empty diagram is in $\mathcal{S}$; $\mathcal{S}$ is closed under finitary extensions (up to isomorphism); $\mathcal{S}$ is closed under chase; and finally we add that, for any finite list of diagrams in $\mathcal{S}$ there exists mutually disjoint isomorphic copies in $\mathcal{S}$ of those diagrams. We say that a collection $\mathcal{M}$ of model diagrams is *chase-complete* if $\mathcal{M}$ is the collection of model diagrams in a chase-complete category of diagrams. By Proposition 2.2.3, $\mathrm{MDiag}_b(\Sigma)$ is chase-complete. With reference to the chase functor construction of [9], the restriction to a small and chase-complete subcategory of diagrams can be done e.g. by building a set $\mathcal{U}$ based on the natural numbers and the syntax of the theory closed under finite lists; and then consider the bounded diagrams whose domains are subsets of $\mathcal{U}$. We leave the details. For reference, we state, then:

Theorem 2.2.4 *Every regular theory $\mathbb{T}$ has a chase-complete category $\mathcal{M}$ of model diagrams. If the signature is discrete, then the theory has a chase-complete set of discrete model diagrams.*

We say that $\mathcal{M}$ is *conservative* for a class $\mathcal{K}$ of sequents if for every sequent σ in $\mathcal{K}$, if σ is true in all diagrams in $\mathcal{M}$ then $\mathbb{T} \vdash \sigma$. (If $\mathcal{K}$ is left implicit it is understood to be all sequents of the fragment of the theory). We say that $\mathcal{M}$ is *strongly conservative* for $\mathcal{K}$ if it is conservative for $\mathcal{K}$, and, moreover, for every $(D, F) \in \mathcal{M}$, the set of $\mathbb{T}_{(D,F)}$-model diagrams the reducts of which are in $\mathcal{M}$ is conservative for $\mathbb{T}_{(D,F)}$.

Lemma 2.2.5 *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}$ a chase-complete set of model diagrams. Then $\mathcal{M}$ is strongly conservative.*

Proof Let $\mathbb{T}$ and (D, F) be given, and let $\phi \vdash_{\mathbf{x}} \exists \mathbf{y}.\, \psi$ be a normal form regular sequent over Σ extended with D as constants. Assume this sequent to be true in all $\mathbb{T}_{(D,F)}$-model diagrams the reducts of which are in $\mathcal{M}$. Replacing every occurrence of a constant from D in ϕ with a fresh variable z, write $\phi = \phi'[\mathbf{d}/\mathbf{z}]$. Let $\mathbf{s}$ be a list of fresh constants, disjoint from D, of the same length as $\mathbf{x}$ such that $\mathrm{Im}(\mathbf{s})$ is finite. Then $\mathrm{Ch}(D \cup \mathrm{Im}(\mathbf{s}), F \cup \{\langle [\mathbf{x}, \mathbf{z} \mid \phi'], \mathbf{s}, \mathbf{d}\rangle\}) \vDash \mathbb{T}_{D,F}$ and $\mathrm{Ch}(D \cup \mathrm{Im}(\mathbf{s}), F \cup \{\langle [\mathbf{x}, \mathbf{z} \mid \phi'], \mathbf{s}, \mathbf{d}\rangle\} \vDash \phi[\mathbf{s}/\mathbf{x}]$, so $\mathrm{Ch}(D \cup \mathrm{Im}(\mathbf{s}), F \cup \{\langle [\mathbf{x}, \mathbf{z} \mid \phi'], \mathbf{s}, \mathbf{d}\rangle\} \vDash \exists \mathbf{y}.\, \psi[\mathbf{s}/\mathbf{x}]$. Whence $\mathbb{T}_{(D \cup \mathrm{Im}(\mathbf{s}), F \cup \{\langle [\mathbf{x}, \mathbf{z} \mid \phi'], \mathbf{s}, \mathbf{d}\rangle\}}$ proves the sequent $\top \vdash \exists \mathbf{y}.\, \psi[\mathbf{s}/\mathbf{x}]$. Then $\mathbb{T}_{(D,F)}$ proves the sequent $\phi[\mathbf{s}/\mathbf{x}] \vdash \exists \mathbf{y}.\, \psi[\mathbf{s}/\mathbf{x}]$. With $\mathrm{Im}(\mathbf{s})$ being a finite set, we can conclude that $\mathbb{T}_{(D,F)}$ proves the sequent $\phi \vdash_{\mathbf{x}} \exists \mathbf{y}.\, \psi$. ⊣

A similar argument shows also that a chase-complete $\mathcal{M}$ is conservative for geometric sequents over the signature of $\mathbb{T}$. We state this for reference.

Lemma 2.2.6 *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}$ a chase-complete set of model diagrams. Then $\mathcal{M}$ is conservative for geometric sequents over the signature of $\mathbb{T}$.*

Proof See [9]. ⊣

Finally, for the statement of Joyal's theorem in Section 3 we transfer the relevant results above to the case of structures for not necessarily purely relational signatures. This is a straightforward application of using the adjoint equivalence between diagrams and structures and of translating between signatures with function symbols and signatures without them, and we display it for reference. For theory $\mathbb{T}$ and model $\mathbf{M}$, the theory $\mathbb{T}_{\mathbf{M}}$ of $\mathbf{M}$ is defined as usual, so that $\mathrm{Mod}(\mathbb{T}_{\mathbf{M}}) \simeq (\mathbf{M} \downarrow \mathrm{Mod}(\mathbb{T}))$.

Corollary 2.2.7 *Let $\mathbb{T}$ be a regular theory over an arbitrary signature Σ (not necessarily purely relational). Then there exists a strongly complete set of models for $\mathbb{T}$.*

2.2.8 Enumerable coherent theories and Fan

The construction of the functor Ch of [9] relied upon in the previous section involves applying all axioms of the theory simultaneously at each step. (In that sense it could be said to be a "simultaneous chase".) In the enumerable setting one can, instead, apply a single axiom in each step. With disjunctions allowed in the axioms, this produces a finitely branching tree of structures, instead of a sequence of structures. Passing from regular to coherent theories, we therefore need the Fan theorem (with decidable bar[5]) to prove completeness. The construction in this case is akin to e.g. [7] (and, as mentioned, the resulting proposition known), and we only outline it. Recall that by positive coherent we mean the coherent fragment without the logical constant $\bot$.

Proposition 2.2.9 (Fan) *Enumerable positive coherent theories are complete with respect to enumerable model diagrams.*

Proof Let $\mathbb{T}$ be a enumerable positive coherent theory over a relational signature Σ, assumed to be axiomatized by sequents on normal form. An *application* of such an axiom

$$\theta \vdash_{\mathbf{x}} \bigvee_{1 \leq i \leq n} \exists \mathbf{y_i}.\, \psi_i$$

to a diagram (D, F) is a function $f : \mathbf{x} \to D$ such that $\langle [\mathbf{x} \mid \theta], f(\mathbf{x}) \rangle \in F$. Such an application induces n children (D_i, F_i), where (D_i, F_i) is the least extension of (D, F) containing a list $\mathbf{y'}_i$ of distinct fresh elements of the same length as $\mathbf{y}_i$ and such that $\psi_i[f(\mathbf{x})/\mathbf{x}, \mathbf{y'}_i/\mathbf{y}_i]$ is true.

Given a finite diagram (D, F), build a finitely branching tree $\mathcal{T}$ of finite diagrams with root (D, F) e.g. as follows. With $\mathbb{T}$ enumerable we can find finite subtheories $\mathbb{T}_n$ such that $\mathbb{T}_0 \subseteq \ldots \mathbb{T}_n \subseteq \ldots \bigcup_{n \in \mathbb{N}} \mathbb{T}_n = \mathbb{T}$. Build a sequence of finite trees $\mathcal{T}_n$ with $\mathcal{T}_0$ consisting just of (D, F) and $\mathcal{T}_n$ an initial

[5] For statement and basic equivalents of the Fan theorem see e.g. [23, Sect. 7], where the relevant principle is named $\mathrm{FAN}_{\mathrm{D}}$.

subtree of $\mathcal{T}_{n+1}$ as follows. For each leaf (D', F') of the tree $\mathcal{T}_n$, list all possible applications $a_1, \ldots, a_m$ of $\mathbb{T}_n$ to that leaf. Apply a_1 to (D', F'). This induces a finite number of children which are extensions of (D', F'). Apply a_2 to those, a_3 to the children induced by that again, and proceed until the list of applications runs out. This produces a finite tree $\mathcal{T}'_{(D',F')}$ with (D', F') as its root. Then $\mathcal{T}_{n+1}$ is obtained by appending $\mathcal{T}'_{(D',F')}$ to each leaf (D', F') of $\mathcal{T}_n$. Let $\mathcal{T}$ be the union of the $\mathcal{T}_n$. Notice that: 1) $\mathcal{T}_n$ constitutes (or translates to) a dynamical cover (c.f. [7]) of (D, F) with respect to $\mathbb{T}$; and 2) the union of the diagrams along a path of $\mathcal{T}$ is a (enumerable) $\mathbb{T}$-model diagram.

Now, let

$$\phi \vdash_{\mathbf{x}} \bigvee_{i \leq n} \exists \mathbf{y_i}.\, \psi_i$$

be a positive coherent sequent (on normal form, without loss of generality). Assume it is true in all $\mathbb{T}$-model diagrams. Let (D, F) be the finite diagram presented by the formula-in-context $\mathbf{x}.\phi$; that is the least diagram for which the domain consists of a finite set bijective with $\mathrm{Im}(\mathbf{x})$, we can write $\bar{\mathbf{x}}$, and $\phi[\bar{\mathbf{x}}/\mathbf{x}]$ is true. Construct the tree $\mathcal{T}$ on (D, F) as above. Define the subset of nodes

$$B = \left\{(D', F') \;\middle|\; \exists i \leq n.\, \exists \mathbf{d} \in D'.\, \psi_i[\bar{\mathbf{x}}/\mathbf{x}, \mathbf{d}/\mathbf{y}_i] \in F'\right\}$$

This is a decidable subset, and since the sequent is true in all models, and thus in all paths, it is a bar. Thus by the Fan theorem, it is a universal bar. Accordingly, there is an n such that every leaf node in $\mathcal{T}_n$ is in B. And since $\mathcal{T}_n$ is a dynamical cover, the sequent is provable in $\mathbb{T}$. $\dashv$

The fallible Kripke semantics of e.g. [27] allows for *exploding* nodes, in the form of nodes that force $\bot$. Such nodes force all other formulas as well. Similarly we could define an exploding structure as one that interprets $\bot$ as true. We prefer to look at this through the lense of Morleyization; a structure for $\mathbb{T}^m$ yields a structure for $\mathbb{T}$ in which some of the logical constants are interpreted non-standardly. In particular, if $\mathbb{T}$ is a coherent theory and $\mathbb{T}^m$ its positive coherent Morleyization, then a $\mathbb{T}^m$-model diagram (D, F) induces an interpretation of the formulas of $\mathbb{T}$, by letting the extension of a formula ϕ be the extension of the corresponding predicate P_ϕ. We say that this interpretation is *exploding* if $P_\bot \in F$. And we refer to the interpretation of $\mathbb{T}$ in terms of the models of $\mathbb{T}^m$ as possibly exploding or *fallible* Tarski semantics. We will consider further modifications in the next section. We now have the following corollary of Proposition 2.2.9.

Corollary 2.2.10 (Fan) *Enumerable coherent theories are complete with respect to fallible Tarski semantics.*

If $\mathbb{T}$ is an enumerable classical first-order theory, then its coherent Morleyization $\mathbb{T}^m$ is an enumerable coherent theory. A model for $\mathbb{T}^m$ can be regarded as an ordinary Tarski model for $\mathbb{T}$ satisfying LEM; that is to say, every set that is the extension of a first-order formula of the language of the theory $\mathbb{T}$ must be complemented. Thus regarding a classical first-order theory as a first-order theory containing the LEM axiom scheme, we have as a consequence:

Corollary 2.2.11 (Fan) *Classical enumerable first-order theories are complete with respect to fallible Tarski semantics.*

Remark 2.2.12 Corollary 2.2.11 provides an intuitionistic completeness theorem for classical logic provided the notion of model is relaxed to allow exploding models. Such theorems have been derived before, notably by Krivine (see [13] and also [4]). In fact, Krivine proves the model existence theorem for consistent classical first-order theories (which is intuitionistically stronger than the completeness theorem), but with respect to models in which disjunction is non-standardly interpreted. In comparison, Corollary 2.2.11 retains the standard semantics for all connectives except $\bot$.

Finally, we show that the use of the Fan theorem in Proposition 2.2.9 is essential, and conclude:

Theorem 2.2.13 *The completeness of enumerable positive coherent theories with respect to enumerable model diagrams is equivalent to the Fan theorem.*

Proof Let F be a fan containing a decidable bar $\mathbf{B}$. We follow the notation in [25, 4.1]. Consider the theory $\mathbb{T}$ over the signature consisting of a propositional variable $P_{\mathbf{n}}$ for each element $\mathbf{n} \in F$ and a propositional variable B, and whose axioms are:

1. $\top \vdash P_{\langle\rangle}$, where $\langle\rangle$ is the root of the Fan;
2. $P_{\mathbf{n}} \vdash \bigvee_{\mathbf{n}*m \in F} P_{\mathbf{n}*m}$;
3. $P_{\mathbf{n}} \vdash B$ for each $\mathbf{n} \in \mathbf{B}$.

For each branch α in F let S_α be the set of sentences

$$S_\alpha = \{P_{\mathbf{n}} \mid \exists x.\, \bar{\alpha}x = \mathbf{n}\}$$

Let $\mathbf{M} = (D, F)$ be a enumerable $\mathbb{T}$-model. Since it is enumerable we can find a branch α such that $S_\alpha \subseteq F$. Since $\mathbf{B}$ is a bar, we then have that $B \in F$. Thus $\mathbf{M} \vDash (\top \vdash B)$. By completeness, there is a proof of $\top \vdash B$ in $\mathbb{T}$, with finitely many axioms, whence $\mathbf{B}$ must be uniform. $\dashv$

Since classical first-order logic is conservative over coherent logic (see e.g. [22]), we could have added the fallible Tarski-completeness of classical FOL as a third equivalent statement in 2.2.13. We proceed now to the theorem of Joyal by which one can add the fallible Kripke completeness of FOL as a fourth. Before doing so, however, we note, for use in Section 4, that if $\mathbb{T}$ is a regular theory then the construction of Proposition 2.2.9 yields a sequence of diagrams, the union of which is a model of $\mathbb{T}$. That this construction, extended to general enumerable diagrams, can be used to show that enumerable $\mathbb{T}$ models are weakly reflective in enumerable diagrams, as in the classical case, is rather expected and straightforward. We therefore state the following for reference and without proof.

Proposition 2.2.14 *Let $\mathbb{T}$ be an enumerable regular theory. There exists a chase-complete set $\mathcal{S}$ of enumerable diagrams with a chase functor* $\mathrm{Ch}: \mathcal{S} \longrightarrow \mathcal{M}$*—where $\mathcal{M}$ is the subcategory of model diagrams—which is moreover a weak reflection. That is, for any homomorphism* $\mathrm{h}: (D, F) \longrightarrow (D', F')$ *in $\mathcal{S}$ where $(D', F') \in \mathcal{M}$ there exists a homomorphism* $\hat{\mathrm{h}}: \mathrm{Ch}(D, F) \longrightarrow (D', F')$ *such that*

$$\begin{tikzcd} \mathrm{Ch}(D,F) \arrow[r, "\hat{h}"] & (D',F') \\ (D,F) \arrow[u, "c_{(D,F)}"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes.

3 Joyal's theorem

Let $\mathbb{T}$ be a coherent (or regular) theory. A coherent (regular) formula-in-context $[\mathbf{x} \mid \phi]$ induces an evaluation functor

$$\mathrm{Ev}_{[\mathbf{x} \mid \phi]} : \mathrm{Mod}(\mathbb{T}) \longrightarrow \mathbf{Set}$$

by $\mathbf{M} \mapsto [\![\mathbf{x}|\phi]\!]^{\mathbf{M}}$. Mapping a formula-in-context to its corresponding evaluation functor defines (by soundness) a functor $\mathrm{Ev} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathrm{Mod}(\mathbb{T})}$, which we also call the evaluation functor, trusting that context will prevent confusion. Since the coherent structure in a presheaf category is computed pointwise, the following is immediate and stated only for emphasis and reference.

Lemma 3.0.1 *Let $\mathbb{T}$ be a coherent (regular) theory and $\mathcal{C}_{\mathbb{T}}$ its coherent (regular) syntactic category. The functor*

$$\mathrm{Ev} \ : \ \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathrm{Mod}(\mathbb{T})}$$

which sends a formula to its corresponding evaluation functor is coherent (regular).

We can now give the "constructive content" of the Kripke completeness theorem of A. Joyal—cf. [18, Thm 6.3.5]—first in the form of the following theorem for regular theories. Since the purpose is to give a constructive restatement of this classical theorem, we state it first in terms of arbitrary signatures and ordinary Tarski models. The proof is not in essence dissimilar from the one in [18]. Recall that by Corollary 2.2.7 there are strongly complete sets of models for regular theories.

Theorem 3.0.2 *Let Σ be a single sorted theory, not restricted in size, nor necessarily discrete (and possibly containing function symbols). Let $\mathbb{T}$ be a regular theory over Σ, and let $\mathcal{M}$ be a full subcategory of* $\mathrm{Mod}(\mathbb{T})$ *such that $\mathcal{M}$ is strongly conservative. Then the functor*

$$\mathrm{Ev}\ :\ \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathcal{M}}$$

is a) conservative and b) whenever the pullback functor $\mathrm{f}^*:\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(B) \longrightarrow \mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(A)$ *induced by a morphism $f:A \longrightarrow B$ in $\mathcal{C}_{\mathbb{T}}$ has a right adjoint $\forall_f$ we have for all $S \in \mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(A)$ that $Ev(\forall_f(S)) = \forall_{Ev(f)}(Ev(S))$.*

Proof a) For formulas $[\mathbf{x} \mid \phi]$ and $[\mathbf{x} \mid \psi]$, if $\mathrm{Ev}_{[\mathbf{x} \mid \phi]}(\mathbf{M}) \subseteq \mathrm{Ev}_{[\mathbf{x} \mid \psi]}(\mathbf{M})$ for all $\mathbf{M} \in \mathcal{M}$ then $\mathbf{M} \vDash (\phi \vdash_{\mathbf{x}} \psi)$ for all $\mathbf{M} \in \mathcal{M}$, whence $\phi \vdash^{\mathbb{T}}_{\mathbf{x}} \psi$ by completeness.

b) The non-trivial direction is $Ev(\forall_f(S)) \supseteq \forall_{Ev(f)}(Ev(S))$. It suffices to consider a situation

$$\begin{tikzcd}
S = [\mathbf{x} \mid \theta] \arrow[d, hook] & & \forall_f(S) = [\mathbf{y} \mid \gamma] \arrow[d, hook] \\
A = [\mathbf{x} \mid \phi] \arrow[rr, "f = |[\mathbf{x},\mathbf{y} \mid \lambda]|"'] & & B = [\mathbf{y} \mid \psi]
\end{tikzcd}$$

in $\mathcal{C}_{\mathbb{T}}$, where $\theta \vdash^{\mathbb{T}}_{\mathbf{x}} \phi$ and $\gamma \vdash^{\mathbb{T}}_{\mathbf{y}} \psi$. Applying the functor Ev and evaluating at a model $\mathbf{M}$ we have

$$\begin{tikzcd}
\{\mathbf{d} \mid \mathbf{M} \vDash \theta(\mathbf{d})\} \arrow[d, hook] & & \{\mathbf{c} \mid \mathbf{M} \vDash \gamma(\mathbf{c})\} \arrow[d, hook] \\
\{\mathbf{d} \mid \mathbf{M} \vDash \phi(\mathbf{d})\} \arrow[rr, "\mathrm{Ev}_f(\mathbf{M}) = \{\mathbf{d},\mathbf{c} \mid \mathbf{M} \vDash \lambda(\mathbf{d},\mathbf{c})\}"'] & & \{\mathbf{c} \mid \mathbf{M} \vDash \psi(\mathbf{c})\}
\end{tikzcd}$$

Let $\mathbf{c} \in \forall_{\mathrm{Ev}_f}(\mathrm{Ev}_{[\mathbf{x} \mid \theta]})(\mathbf{M}) \subseteq \mathrm{Ev}_{[\mathbf{y} \mid \psi]}(\mathbf{M})$. Accordingly, for all $g:\mathbf{M} \longrightarrow \mathbf{N}$ in $\mathcal{M}$ we have:

$$\mathrm{Ev}_f((\mathbf{N}))^{-1}\,(\mathrm{Ev}_{[\mathbf{y} \mid \psi]}(g)(\mathbf{c})) \subseteq \mathrm{Ev}_{[\mathbf{x} \mid \theta]}(\mathbf{N}). \tag{1}$$

We show $\mathbf{M} \vDash \gamma(\mathbf{c})$. Let $g:\mathbf{M} \longrightarrow \mathbf{N}$ be a morphism in $\mathcal{M}$, with $\mathbf{N}'$ the corresponding $\mathbb{T}_{\mathbf{M}}$-model. By (1) we have

$$\mathbf{N}' \vDash (\lambda[\mathbf{c}/\mathbf{y}] \vdash_{\mathbf{x}} \theta) \tag{2}$$

Thus, the sequent (2) is true in all $\mathbb{T}_{\mathbf{M}}$-models corresponding to homomorphisms from $\mathbf{M}$ in $\mathcal{M}$, and therefore provable in $\mathbb{T}_{\mathbf{M}}$, by the assumption of strong completeness. By Lemma 2.1.3, there is a regular formula ξ in context $\mathbf{y}$ such that $\mathbf{M} \vDash \xi(\mathbf{c})$ and $\mathbb{T}$ proves the sequent $(\xi \wedge \lambda \vdash_{\mathbf{x},\mathbf{y}} \theta)$. But then, since $\forall_f([\mathbf{x} \mid \theta]) = [\mathbf{y} \mid \gamma]$, we have that $\mathbb{T}$ proves the sequent $(\xi \wedge \psi \vdash_{\mathbf{y}} \gamma)$. Whence $\mathbf{M} \vDash \gamma(\mathbf{c})$. $\dashv$

It is convenient to have a name for the property proved in Theorem 3.0.2. Following e.g. [5] (at least for the first notion):

Definition 3.0.3 We say that a functor $\mathrm{F}:\mathcal{C} \longrightarrow \mathcal{D}$ from a coherent category to a Heyting category is *conditionally Heyting* if it is coherent and preserves any right adjoints to pullback functors that might exist in $\mathcal{C}$. If F and $\mathcal{C}$ are regular, we say F is *conditionally sub-Heyting* if it is regular and preserves any right adjoints to pullback functors that might exist in $\mathcal{C}$.

The equivalence $\mathrm{Mod}(\mathbb{T}) \simeq \mathrm{MDiag}(\mathbb{T})$ induces an equivalence $\mathbf{Set}^{\mathrm{Mod}(\mathbb{T})} \simeq \mathbf{Set}^{\mathrm{MDiag}(\mathbb{T})}$. Accordingly, if $\mathcal{M}$ is a strongly complete set of model diagrams, the composite

$$\mathcal{C}_{\mathbb{T}} \xrightarrow{\mathrm{Ev}} \mathbf{Set}^{q(\mathcal{M})} \simeq \mathbf{Set}^{\mathcal{M}}$$

is conservative and conditionally sub-Heyting. When returning to working with diagrams in the sequel, we shall consider this functor, also under the name Ev.

As a corollary Theorem 3.0.2 we have the following version for coherent theories. Strongly complete sets of models do not in general exist for coherent theories. By Proposition2.2.9, however, they do for enumerable positive coherent theories under the assumption of the Fan theorem.

Corollary 3.0.4 *Let $\mathbb{T}$ be a coherent theory, with $\mathcal{C}_{\mathbb{T}}$ its coherent syntactic category, and suppose $\mathcal{M}$ is a strongly conservative category of $\mathbb{T}$-model diagrams. Then the evaluation functor*

$$\mathrm{Ev} \ : \ \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathcal{M}}$$

is a conservative and conditionally Heyting functor.

Proof The proof of 3.0.2 can be repeated for this case. Alternatively, consider the regular Morleyization $\mathbb{T}^m$ of the coherent theory $\mathbb{T}$. Notice, first, that $\mathcal{M}$ can then be considered as a full subcategory of $\mathrm{Mod}(\mathbb{T}^m)$, and as such it is then strongly conservative for $\mathbb{T}^m$. Then notice that evaluation restricted to $\mathcal{M}$ yields a coherent functor from $\mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^m}$ to $\mathbf{Set}^{\mathcal{M}}$. $\dashv$

4 Sheaf completeness

4.1 Modified completeness

Loosely and informally, let us say that a model is *modified* if some connectives are interpreted in a non-standard way, and *standard* otherwise. We say that it is *fallible* if the only connective treated non-standardly is $\bot$. It is a corollary of Theorem 2.2.4 and Lemma 2.2.5 that regular$_\bot$ theories are complete with respect to fallible Tarski semantics. Now, if $\mathbb{T}$ is, say, a regular$_\bot$ theory, $\mathbb{T}^m$ it's regular Morleyization, and $\mathcal{M}$ is a small, full, and strongly conservative subcategory of $\mathrm{MDiag}(\mathbb{T}^m)$, we have that the evaluation functor

$$\mathrm{Ev} \;:\; \mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^m} \longrightarrow \mathbf{Set}^{\mathcal{M}}$$

is regular (in particular). But it does not preserve the initial object, as $\mathrm{Ev}_{[\,|\,P_\bot]}(-)$ is not the constant empty functor 0. Thus it can be viewed as a conservative fallible presheaf model of $\mathbb{T}$. We shall obtain a conservative standard sheaf model by taking sheaves with respect to the least coverage (on $\mathcal{M}^{\mathrm{op}}$) so that $\mathrm{Ev}_{[\,|\,P_\bot]}(-)$ is identified with 0. Accordingly, we obtain a model of $\mathbb{T}$ in a closed subtopos (in the sense of [11, A4.5.3]) of $\mathbf{Set}^{\mathcal{M}}$. Similarly, if $\mathbb{T}$ is a coherent theory and $\mathbb{T}^m$ its regular Morleyization, $\mathrm{Ev} \;:\; \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathcal{M}}$ does not preserve finite disjunctions. A conservative standard model will be obtained by sheafifying with respect to the least coverage such that finite disjunctions are preserved. A conservative model will also be given by a slightly stronger coverage given (in part) in terms of binary trees and which is, in that sense, akin to a (fallible) Beth model. Classically, or in a enumerable setting, the latter two coverages are equivalent. In a countable setting, they also give rise to a Beth-completeness theorem, establishing a link between the least coverage forcing a standard interpretation and Beth semantics.

The coverages are given in terms of sieves on $\mathcal{M}^{\mathrm{op}}$, and thus cosieves on $\mathcal{M}$. Explicitly, then, let $\mathbb{T}$ be a theory in a fragment with $\bot$, $\mathbb{T}^m$ its regular Morleyization, and $\mathcal{M}$ a small full subcategory of $\mathrm{MDiag}(\mathbb{T}^m)$. Let the *exploding coverage* E be the coverage which assigns to each $(D, F) \in \mathcal{M}$ the set of cosieves $E(D, F) = \{\emptyset \mid P_\bot \in F\}$. This is a coverage (in the sense of [11, A2.1.9, C2.1.1]) since if $S \in E(D_1, F_1)$ and $f : (D_1, F_1) \longrightarrow (D_2, F_2)$ is a homomorphism, then $S = \emptyset$ and $P_\bot \in F_1$, whence $P_\bot \in F_2$, since it is preserved by f, so $\emptyset \in E(D_2, F_2)$. We then have the following addition to Theorem 3.0.2.

Proposition 4.1.1 *Let $\mathbb{T}$ be an (at least) regular$_\bot$ theory, $\mathbb{T}^m$ its regular Morleyization, $\mathcal{M}$ be a strongly conservative, small, full subcategory of* $\mathrm{MDiag}(\mathbb{T}^m)$ *and E be the exploding coverage. Then the evaluation functor* *factors through sheaves*

$$\begin{tikzcd} \mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^m} \arrow[r, "\mathrm{Ev}"] \arrow[dr] & \mathbf{Set}^{\mathcal{M}} \\ & \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E) \arrow[u, hook] \end{tikzcd}$$

So that $\mathrm{Ev} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E)$ *is conservative, conditionally sub-Heyting, and preserves the initial object.*

Proof By Theorem 3.0.2 it remains only to show that Ev factors through $\mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E)$ and that $\mathrm{Ev}_{[\,\mid\,\perp]}(-)$ is terminal in $\mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E)$. First, $\mathrm{Ev}_{[\mathbf{x}\,\mid\,\phi]}(-)$ is a sheaf since if $S \in E(D,F)$, then S is empty and (D,F) is exploding, which means that $\mathrm{Ev}_{[\mathbf{x}\,\mid\,\phi]}(D,F) = \{*\}$. Second, $\mathrm{Ev}_{[\,\mid\,\perp]}(D,F) = \{* \mid P_{\perp} \in F\}$ which is the initial sheaf in $\mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E)$. ⊣

Note that classically the standard (i.e. non-exploding) models in $\mathcal{M}$ are dense (in the sense of [11]), so that then $\mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, E) \simeq \mathbf{Set}^{\mathcal{M}^s}$ where $\mathcal{M}^s$ is the full subcategory of standard models.

Next, let $\mathbb{T}$ be a theory in a fragment with $\vee$ and $\perp$, with $\mathbb{T}^m$ its regular Morleyization and $\mathcal{M}$ a small full subcategory of $\mathrm{MDiag}(\mathbb{T}^m)$. Again,

$$\mathrm{Ev} \ : \ \mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^m} \longrightarrow \mathbf{Set}^{\mathcal{M}}$$

is regular and conservative, but fails to preserve $\vee$ as well as $\perp$. Again we make explicit the least coverage B forcing a standard interpretation. That is, the least coverage such that the initial object 0 is dense in $\mathrm{Ev}_{[\,\mid\,P_{\perp}]}(-)$ and, for all disjunctions $[\mathbf{x} \mid \phi \vee \psi]$ of $\mathbb{T}$, $\mathrm{Ev}_{[\mathbf{x}\,\mid\,P_{\phi}]}(-) \vee \mathrm{Ev}_{[\mathbf{x}\,\mid\,P_{\psi}]}(-)$ is dense in $\mathrm{Ev}_{[\mathbf{x}\,\mid\,P_{\phi\vee\psi}]}(-)$. First, for all disjunctions(-in-context) $[\mathbf{x} \mid \phi \vee \psi]$ of $\mathbb{T}$, model diagrams (D,F) in $\mathcal{M}$, and lists of elements $\mathbf{d} \in D^{l(\mathbf{x})}$, let $S_{\langle[\mathbf{x}\,\mid\,P_{\phi\vee\psi}],\mathbf{d}\rangle}$ be the following cosieve on (D,F):

$$S_{\langle[\mathbf{x}\,\mid\,P_{\phi\vee\psi}],\mathbf{d}\rangle} = \big\{\mathrm{h}\colon (D,F) \longrightarrow (D',F') \ \big|\ (D',F') \vDash P_{\phi}[h(\mathbf{d})/\mathbf{x}] \vee P_{\psi}[h(\mathbf{d})/\mathbf{x}]\big\}$$

Then let B be specified by

$$B(D,F) = E(D,F) \cup \left\{ S_{\langle[\mathbf{x}\,\mid\,P_{\phi\vee\psi}]\rangle,\mathbf{d}} \ \middle|\ \langle[\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle \in F \right\}$$

Again, B is a coverage. We refer to it as the *minimal coverage.* A connection to Beth semantics will be displayed in Section 4.1.14. The proof of the following is similar to that of Theorem 4.1.4, and a corollary of it if $\mathcal{M}$ is chase-complete, and is therefore omitted.

Proposition 4.1.2 *Let* $\mathbb{T}$ *be an (at least) coherent theory,* $\mathbb{T}^m$ *its regular Morleyization,* $\mathcal{M}$ *a strongly conservative, small, full subcategory of*

MDiag($\mathbb{T}^m$), *and B be the minimal coverage on $\mathcal{M}^{\mathrm{op}}$. Then the evaluation functor* Ev *composed with the sheafification functor* $\mathrm{a}\colon \mathbf{Set}^{\mathcal{M}} \longrightarrow \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, B)$

$$\begin{tikzcd} \mathcal{C}_{\mathbb{T}} \arrow[r, "\mathrm{Ev}"] \arrow[dr, "a\circ\mathrm{Ev}"'] & \mathbf{Set}^{\mathcal{M}} \arrow[d, shift right, "a"'] \\ & \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, B) \arrow[u, hook, shift right] \end{tikzcd}$$

is conservative, coherent, and conditionally Heyting.

If $\mathcal{M}$ is chase-complete, then the minimal coverage on $\mathcal{M}^{\mathrm{op}}$ can be strengthened while still yielding a conservative $\mathbb{T}$-model as follows. Let $(D, F) \in \mathcal{M}$, let $[\mathbf{x} \mid \phi \vee \psi]$ be a disjunction of $\mathbb{T}$, and let $\mathbf{d} \in D$ such that $\langle [\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle \in F$. Then we have,

$$\begin{tikzcd} \mathrm{Ch}(D, F \cup \{P_\phi[\mathbf{d}]\}) & & \mathrm{Ch}(D, F \cup \{P_\psi[\mathbf{d}]\}) \\ & (D, F) \arrow[ul, "c_0"] \arrow[ur, "c_1"'] & \end{tikzcd}$$

with c_0 the homomorphism induced by $(D, F) \subseteq (D, F \cup \{P_\phi[\mathbf{d}]\}) \subseteq \mathrm{Ch}(D, F \cup \{P_\phi[\mathbf{d}]\})$, and similarly for c_1. We refer to such a pair, given by a fact of the form $\langle [\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle$ in F, as a *chase pair* over (D, F). Since Ch is a functor, assigning to each $(D, F) \in \mathcal{M}$ the set of chase pairs over it is a coverage. To this we also add the coverage E, so that the family of covering families over (D, F) is the union of the set of chase pairs and the set $\{\emptyset \mid (D, F) \vDash P_\bot\}$. Denote the resulting coverage by C, and the least Grothendieck coverage containing C by $\overline{C}$. Similarly, write $\overline{B}$ for the least Grothendieck coverage containing B. We refer to C as the *disjunctive* coverage. The two coverages compare as follows.

Lemma 4.1.3 *Let $\mathbb{T}$ be a coherent theory, $\mathbb{T}^m$ its regular Morleyization, and $\mathcal{M}$ a chase-complete category of model diagrams for $\mathbb{T}^m$. Let $\overline{B}$ and $\overline{C}$ be the least Grothendieck coverages containing the minimal and disjunctive coverages on $\mathcal{M}^{\mathrm{op}}$, respectively. Then $\overline{B} \subseteq \overline{C}$. Moreover,*

1. *If* Ch *is a weak reflection then also $\overline{C} \subseteq \overline{B}$.*
2. *The statement that $\overline{C} = \overline{B}$ for arbitrary $\mathbb{T}$ is equivalent to the Axiom of Choice.*

Proof (1) and that $\overline{C} \supseteq \overline{B}$ is clear. Since (2) can be considered more of a remark that will play no further role for us here, we only outline the proof:

Consider the *coherent* theory $\mathbb{T}$ with unary predicate symbols B and $\bar{B}$, one binary relation symbol R, and the axioms

$$B(x) \wedge \bar{B}(x) \vdash_x \bot \qquad B(x) \vdash_x \exists y.\, R(x, y) \qquad \dashv$$

A surjection $e : Y \twoheadrightarrow X$ can be considered as a $\mathbb{T}$-model, and therefore a $\mathbb{T}^m$-model $\mathbf{E}$ by, briefly, letting $|\mathbf{E}|$ be $X+Y$, X be the extension of B, and the extension of R be the inverse of the graph of e.

Write $\beta^{\mathbf{x}} := \bigwedge_{x\in\mathbf{x}}(B(x) \vee \bar{B}(x))$. Let (D,F) be the Σ^m-diagram presented by $D = \{*\}$ and
$F = \left\{\langle[\mathbf{x} \mid P_\phi], \vec{*}\rangle \;\middle|\; [\mathbf{x} \mid \phi] \text{ coherent in canonical context, and } \beta^{\mathbf{x}} \vdash^{\mathbb{T}}_{\mathbf{x}} \phi\right\}$.
Then (D,F) is a (finite) $\mathbb{T}^m$-model diagram.

Now, if $\overline{C} = \overline{B}$ then the family $S_{\langle[x \mid P_{B(x)\vee\bar{B}(x)}],*\rangle}$ is the sieve generated by the chase pair given by $\langle[x \mid P_{B(x)\vee\bar{B}(x)}],*\rangle$. Then, for a surjection $e : Y \twoheadrightarrow X$, elements in X induces homomorphisms $(D,F) \twoheadrightarrow \mathbf{E}$, and the lifting through $\mathrm{Ch}(D, F \cup \{P_{B(x)}[\,*\,]\})$ gives a splitting of e (see the proof of Theorem 5.16 in [9]).

Theorem 4.1.4 *Let $\mathbb{T}$ be an at least coherent theory, $\mathbb{T}^m$ its regular Morleyization, $\mathcal{M}$ a full, chase-complete subcategory of* $\mathrm{MDiag}(\mathbb{T}^m)$*, and C be the disjunctive coverage on $\mathcal{M}^{\mathrm{op}}$. Then the evaluation functor* Ev *composed with the sheafification functor* $\mathrm{a}\colon \mathbf{Set}^{\mathcal{M}} \longrightarrow \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, C)$

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{T}} \arrow[r, "\mathrm{Ev}"] \arrow[dr, "a\circ\mathrm{Ev}"'] & \mathbf{Set}^{\mathcal{M}} \arrow[d, shift right, "a"'] \\
& \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, C) \arrow[u, hook, shift right]
\end{tikzcd}$$

is conservative, coherent, and conditionally Heyting.

Proof Let $U^n = \mathrm{Ev}_{[\mathbf{x} \mid \top]}(-)$ where the length of $\mathbf{x}$ is n. Recall from e.g. [15] that the (coherent) sheafification functor a restricts to (one half of) a poset isomorphism between closed subobjects of U^n and subobjects of $a(U^n)$,

$$\mathrm{ClSub}(U^n) \cong \mathrm{Sub}(a(U^n))$$

It follows that it is sufficient to show that, for any formula $[\mathbf{x} \mid \theta]$ of $\mathbb{T}$, the functor $\mathrm{Ev}_{[\mathbf{x} \mid P_\theta]}(-)$ is C-closed in the subobject lattice of U^n. Let $(D,F) \in \mathcal{M}$, let $[\mathbf{y} \mid \phi \vee \psi]$ be a disjunction of $\mathbb{T}$, and let $\mathbf{d} \in D^{\mathrm{length}(\mathbf{y})}$ such that $(D,F) \vDash P_{\phi\vee\psi}[\mathbf{d}/\mathbf{y}]$. Let $(D_1,F_1) = \mathrm{Ch}(D, F \cup \{P_\phi[\mathbf{d}]\}))$ and $(D_2,F_2) = \mathrm{Ch}(D, F \cup \{P_\psi[\mathbf{d}]\}))$. Let $\mathbf{c} \in D^n$ and assume $(D_1,F_1),(D_2,F_2) \vDash P_\theta[\mathbf{c}/\mathbf{x}]$. Then $\mathbb{T}^m_{(D,F)}$ proves the sequents $P_\phi[\mathbf{d}/\mathbf{y}] \vdash P_\theta[\mathbf{c}/\mathbf{x}]$ and $P_\psi[\mathbf{d}/\mathbf{y}] \vdash P_\theta[\mathbf{c}/\mathbf{x}]$. Thus there exists proofs with premisses in $\mathbb{T}^m$ of $\chi \wedge P_\phi[\mathbf{d}/\mathbf{y}] \vdash P_\theta[\mathbf{c}/\mathbf{x}]$ and $\xi \wedge P_\psi[\mathbf{d}/\mathbf{y}] \vdash P_\theta[\mathbf{c}/\mathbf{x}]$ where χ and ξ are conjunctions of atomic sentences over $\Sigma \cup D$ which are true in (D,F). By Lemma 2.1.2, there are proofs with the same premisses of sequents $\chi' \wedge P'_\phi \vdash_{\mathbf{w}} P'_\theta$ and $\xi' \wedge P''_\psi \vdash_{\mathbf{v}} P''_\theta$ over Σ, where we assume $\mathbf{w}$ and $\mathbf{v}$ disjoint, and a function $\mathrm{f}\colon \mathbf{w},\mathbf{v} \longrightarrow D$ such that $\chi'[f] = \chi$ and $\xi'[f] = \xi$ and $P'_\phi[f] = P_\phi[\mathbf{d}/\mathbf{y}]$ and $P''_\psi[f] = P_\psi[\mathbf{d}/\mathbf{y}]$ and $P'_\theta[f] = P''_\theta[f] = P_\theta[\mathbf{c}/\mathbf{x}]$. Form the finite set E of equalities with variables from the (assumed disjoint) lists $\mathbf{w}$,$\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ as follows: for each argument slot in $\mathbf{P}_\phi$, with that slot occupied by, say, y in P_ϕ, w in P'_ϕ, and v in P''_ϕ, add $y = w$ and $y = v$ to E; for each argument slot in $\mathbf{P}_\psi$, with that slot occupied by, say, y in P_ψ, w in P'_ψ, and v in P''_ψ, add $y = w$ and $y = v$ to E; and for each argument slot in $\mathbf{P}_\theta$, with that slot occupied by, say, x in P_θ, w in P'_θ, and v in P''_θ, add $x = w$ and $x = v$ to E. Let ρ be the conjunction of the equalities in E. Extend f by: if x occurs in an equality in ρ, say $x = w$, then add $x \mapsto f(w)$; and if y occurs in an equality in ρ, say $y = w$, then add $y \mapsto f(w)$. Notice that this is well defined and $P_\phi[f] = P_\phi[\mathbf{d}/\mathbf{y}]$, $P_\psi[f] = P_\psi[\mathbf{d}/\mathbf{y}]$, and $P_\theta[f] = P_\theta[\mathbf{c}/\mathbf{x}]$. Then: 1) $\rho[f]$ is true in (D, F) and 2) the sequents $\exists \mathbf{w}, \mathbf{v}.\, \rho \wedge \chi' \wedge \xi' \wedge P_\phi \vdash_{\mathbf{x},\mathbf{y}} P_\theta$ and $\exists \mathbf{w}, \mathbf{v}.\, \rho \wedge \chi' \wedge \xi' \wedge P_\psi \vdash_{\mathbf{x},\mathbf{y}} P_\theta$ are provable in $\mathbb{T}^m$. We can replace the regular formula $\exists \mathbf{w}, \mathbf{v}.\, \rho \wedge \chi' \wedge \xi'$ by an atomic formula in Σ^m; say P_γ. We then have that $\mathbb{T}$ proves the sequents $\gamma \wedge \phi \vdash_{\mathbf{x},\mathbf{y}} \theta$ and $\gamma \wedge \psi \vdash_{\mathbf{x},\mathbf{y}} \theta$. Thus $\mathbb{T}$ proves $\gamma \wedge (\phi \vee \psi) \vdash_{\mathbf{x},\mathbf{y}} \theta$, whence $\mathbb{T}^m$ proves $P_\gamma \wedge P_{\phi\vee\psi} \vdash_{\mathbf{x},\mathbf{y}} P_\theta$. Since substituting $\mathbf{d}$ for $\mathbf{y}$ and $\mathbf{c}$ for $\mathbf{x}$ makes the antecedent true in the $\mathbb{T}^m$-model (D, F), $P_\theta[\mathbf{c}/\mathbf{x}]$ must also be true in (D, F). $\dashv$

4.1.5 Kripke and generalized Beth models

The models of 4.1.1 and 4.1.4 can be translated to models in presheaves and sheaves on posets using e.g. the Diaconescu cover (see e.g. [15] for a description of the Diaconescu cover). However, in our current setting, we can use, more directly, the poset of model diagrams and inclusions. We state this also as covering lemma, the technical heart of which is the following. Write $\mathrm{MDiag}^{\subseteq}(\mathbb{T})$ for the poset of of model diagrams and inclusions. Write $\pi : \mathrm{MDiag}^{\subseteq}(\mathbb{T}) \longrightarrow \mathrm{MDiag}(\mathbb{T})$ for the functor that sends an inclusion to the homomorphism it induces.

Lemma 4.1.6 *Let* $\mathrm{MDiag}(\mathbb{T})$ *be the category of model diagrams for some (regular or coherent) theory* $\mathbb{T}$. *For any homomorphism* $\mathrm{h} : (D_0, F_0) \longrightarrow (D_1, F_1)$ *where* D_0 *and* D_1 *are disjoint there exists an extension* $a : (D_0, F_0) \subseteq (D_2, F_2)$ *and homomorphisms* r *and* i *as in the following diagram*

$$\begin{tikzcd}
 & (D_1, F_1) \arrow[d, shift left, "i"] \\
(D_0, F_0) \arrow[ur, "h"] \arrow[r, "\pi(a)"'] & (D_2, F_2) \arrow[u, shift left, "r"]
\end{tikzcd}$$

such that the outer triangle commutes and $r \circ i$ *is the identity on* (D_1, F_1).

PROOF Let $D_2 = D_0 \cup D_1$ and let F_2 be the diagram generated by $F_0 \cup F_1 \cup \{d = d' \mid h(d, d')\}$. We have $a : (D_0, F_0) \subseteq (D_2, F_2)$. Let i be the homomorphism induced by the inclusion $(D_1, F_1) \subseteq (D_2, F_2)$. Then

$$\pi(a)(d, d') \Leftrightarrow \exists c \in D_1 .\, h(d, c) \wedge i(c, d')$$

so the outer triangle commutes. Let $r(d', d) \Leftrightarrow (d' = d) \in F_1 \vee h(d', d)$. It is then straightforward that r is a well-defined homomorphism, as well as both a left and right inverse to i, from which it also follows that (D_2, F_2) is a $\mathbb{T}$-model diagram. $\dashv$

Clearly, the assumption that the domain and codomain of h are disjoint can be done without loss if the codomain can be replaced by an isomorphic copy disjoint from both it and the domain. If $\mathcal{M}$ is a chase-complete set of model diagrams for a regular theory $\mathbb{T}$ and $\mathcal{M}^{\subseteq}$ is the poset of model diagrams in $\mathcal{M}$ and inclusions, then Lemma 4.1.6 implies (by e.g. [11, C3.1.2]) that the restriction functor

$$\mathbf{Set}^{\mathcal{M}} \xrightarrow{\pi^*} \mathbf{Set}^{\mathcal{M}^{\subseteq}}$$

is Heyting and conservative. By Theorem 3.0.2 we then obtain the following new version of Joyal's theorem:

Theorem 4.1.7 *Let $\mathbb{T}$ be a regular theory and $\mathcal{M}^{\subseteq}$ a chase-complete set of model diagrams ordered by inclusions. Then the evaluation functor*

$$\mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Set}^{\mathcal{M}^{\subseteq}}$$

is conservative and conditionally sub-Heyting.

We have, as corollaries to Theorem 3.0.2 (or Proposition 4.1.1), (fallible) Kripke completeness results for theories in certain fragments of first-order logic. By 4.1.6, the underlying poset of the Kripke models can be taken to be a set of model diagrams for the regular Morleyization of $\mathbb{T}$ ordered by inclusion. Similarly, as a corollary of Theorem 4.1.4 we have a completeness theorem for first-order theories with respect to a generalized version of Beth semantics. These are fairly straighforward cases of translating models on presheaves and sheaves on posets to Kripke and Beth-style presentations. For explicitness, we give some further details, also making clear what notions of Kripke and "generalized" Beth models we have in mind:

Let Σ be a relational signature. Write $\mathscr{S}$ for the partially ordered class of Σ-diagrams and (homomorphic) inclusions. Thus an object S in $\mathscr{S}$ is a Tarski structure for Σ with a congruence relation $[\![=]\!]^S$ interpreting $=$. Write $\mathscr{F}$ for the partially ordered class of *fallible* Σ-diagrams and diagram inclusions: an object S in $\mathscr{F}$ is an inhabited diagram for Σ with a subset $[\![\bot]\!]^S \subseteq 1$ of the terminal set interpreting $\bot$, and such that S satisfies the axioms $\bot \vdash_{\mathbf{x}} \phi$ for all atomic formulas ϕ in canonical context $\mathbf{x}$ over Σ. The inclusions in $\mathscr{F}$ must preserve $[\![\bot]\!]$, i.e. $S \subseteq S' \Rightarrow [\![\bot]\!]^S \subseteq [\![\bot]\!]^{S'}$.

Definition 4.1.8 (I) Let Σ be a relational signature. By a *generalized* (*fallible*) *Beth structure* for Σ we mean a triple $\langle P, D, T\rangle$ where P is a poset; $\mathfrak{D}$ is a functor from P to $\mathscr{S}$ ($\mathscr{F}$); and T is an assignment of inhabited sets of subposets of P to nodes of P such that:

(i) all elements of $T(p)$ are finite, binary trees with root p, and $T(p)$ is closed under initial binary subtrees;

(ii) if $t \in T(p)$ with leaf nodes $q_1, \ldots, q_n$ and $t_1 \in T(q_1), \ldots, q_n \in T(q_n)$ then the tree obtained by extending t with the t_i's is in $T(p)$; and for all $p \in P$, $t \in T(p)$, and $q \in t$, $t \cap \uparrow q \in T(q)$; and

(iii) for all $p \leq p'$ in P and $t \in T(p)$ there exists $t' \in T(p')$ such that for all leaf nodes q' of t' there exists a leaf node q of t such that $q \leq q'$.

The clauses of the forcing relation $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ between $p \in P$, first-order formulas-in-context $\mathbf{x}.\phi$, and $\mathbf{d} \in \mathfrak{D}(p)^{l(\mathbf{x})}$ are then:

(a) for ϕ atomic or equal $\bot$ or $\top$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if there exists $t \in T(p)$ such that for all leaf nodes $q \in t$ it is the case that $\mathfrak{D}(q) \vDash \phi[\mathbf{d}/\mathbf{x}]$;

(b) for $\phi = \psi \wedge \theta$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if $p \Vdash \psi[\mathbf{d}/\mathbf{x}]$ and $p \Vdash \theta[\mathbf{d}/\mathbf{x}]$;

(c) for $\phi = \psi \vee \theta$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if there exists $t \in T(p)$ such that for all leaf nodes q in t it is the case that $q \Vdash \psi[\mathbf{d}/\mathbf{x}]$ or $q \Vdash \theta[\mathbf{d}/\mathbf{x}]$;

(d) for $\phi = \psi \to \theta$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if for all $p' \geq p$ it is the case that if $p' \Vdash \psi[\mathbf{d}/\mathbf{x}]$ then $p' \Vdash \theta[\mathbf{d}/\mathbf{x}]$;

(e) for $\phi = \exists y. \psi$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if there exists $t \in T(p)$ such that for all leaf nodes q in t there exists $c \in \mathfrak{D}(q)$ such that $q \Vdash \psi[c/y, \mathbf{d}/\mathbf{x}]$; and

(f) for $\phi = \forall y. \psi$, $p \Vdash \phi[\mathbf{d}/\mathbf{x}]$ if for all $p' \geq p$ and all $c \in \mathfrak{D}(p')$ it is the case that $q \Vdash \psi[c/y, \mathbf{d}/\mathbf{x}]$.

(II) By a (*fallible*) *Kripke structure* we mean a generalized (fallible) Beth structure where $T(p)$ contains only the one node tree on p.

(III) By (*fallible*) *Beth structure* we mean a generalized (fallible) Beth structure where P is a binary tree and $T(p)$ is the set of initial binary subtrees of $\uparrow (p)$. (This notion of Beth structure is, then, with respect to the strong rather than the weak notion of forcing, cf. [26, Ch.13 1.8].)

(IV) By a (*generalized, fallible*) *Beth** *structure* we mean a (generalized, fallible) Beth structure where "covers are only relevant for disjunctions", i.e. one satisfying the following additional conditions:

(1) for ϕ atomic or $\bot$, it is the case that $p \Vdash \phi[\mathbf{d}]$ iff $p \vDash \phi[\mathbf{d}]$, and

(2) for all formulas of the form $\exists x. \phi \in \mathcal{L}$, it is the case that $p \Vdash \exists x. \phi[\mathbf{d}]$ iff there exists $a \in p$ such that $p \Vdash \phi[a, \mathbf{d}]$.

We state corollaries of Theorem 4.1.7, Proposition 4.1.1, and Theorem 4.1.4 in terms of Definition 4.1.8. Let the $\vee$*-free* fragment of FOL be the fragment consisting of sequents not mentioning the connective $\vee$, and the $\bot$*,*$\vee$*-free* fragment be the one not mentioning $\bot$ or $\vee$.

Corollary 4.1.9 *Let $\mathbb{T}$ be a theory in the $\bot$,$\vee$-free fragment over the signature Σ. Then there exists a Kripke model for $\mathbb{T}$ which is conservative (with respect to the $\bot$,$\vee$-free fragment).*

Proof Let $\mathbb{T}^m$ over Σ^m be the regular Morleyization of $\mathbb{T}$. By Theorem 2.2.4 there exists a chase-complete category $\mathcal{M}$ of model diagrams for $\mathbb{T}^m$. By Theorem 4.1.7, the evaluation functor $\mathrm{Ev} : \mathcal{C}_{\mathbb{T}} \simeq \mathcal{C}_{\mathbb{T}^m} \longrightarrow \mathbf{Set}^{\mathcal{M}^{\subseteq}}$ is conditonally sub-Heyting, thus giving a conservative (w.r.t. the $\bot$,$\vee$-free fragment) model of $\mathbb{T}$. Define a Kripke structure K by letting the poset P be $\mathcal{M}^{\subseteq}$ and the functor $\mathfrak{D} : \mathcal{M}^{\subseteq} \longrightarrow \mathscr{S}$ be the forgetfull functor. Let $\mathbf{x}.\phi$ be $\bot$,$\vee$-free over Σ, $S \in \mathcal{M}^{\subseteq}$, and $\mathbf{d}$ a list of the same length as $\mathbf{x}$ of elements in the domain of S . Using that $S \vDash P_\phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow [\mathbf{d}] \in \mathrm{Ev}_{\mathbf{x}.\phi}(S)$ and that Ev is conditionally sub-Heyting, a straightforward induction argument on $\mathbf{x}.\phi$ shows that $S \Vdash^K \phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow S \vDash P_\phi[\mathbf{d}/\mathbf{x}]$.

Corollary 4.1.10 *Let $\mathbb{T}$ be a $\vee$-free theory. Then there exists a fallible Kripke model for $\mathbb{T}$ which is conservative with respect to the* $\vee$-free *fragment.*

Proof From Proposition 4.1.1 and Theorem 4.1.7 (similarly to 4.1.9).

Remark 4.1.11 The restrictions are essential. The existence of a conservative Kripke model (that is, a non-fallible one) for $\vee$-free theories implies LEM (see [19])[6]. A Kripke completeness theorem for $\bot$-free theories, or a fallible Kripke completeness theorem for full FOL, would imply e.g. that the Boolean Prime Ideal theorem is provable in ZF. For theories whose axioms do not mention $\bot$ or $\vee$, such as the empty theory, the existence of a Kripke model which is conservative with respect to all first-order sequents implies MP (see [19]).

As an example application of Theorem 4.1.7, we give a short, semantic proof of the disjunction property for (arbitrary) $\vee$-free theories (cf. [24]) by reducing it to the disjunction property for regular theories (see e.g. [11]; note that the disjunction property for regular theories also directly follows from Proposition 2.2.3).

Corollary 4.1.12 *Let $\mathbb{T}$ be a first-order theory the axioms of which are $\vee$-free, and let ϕ, ψ, and θ be $\vee$-free formulas. If $\mathbb{T}$ proves the sequent $\phi \vdash_{\mathbf{x}} \psi \vee \theta$, then $\mathbb{T}$ proves either $\phi \vdash_{\mathbf{x}} \psi$ or $\phi \vdash_{\mathbf{x}} \theta$.*

[6] In fact, it is equivalent to it, since with LEM we can distinguish between exploding and non-exploding models.

Proof Consider the fallible Kripke model of Corollary 4.1.10. (As in 4.1.9) the nodes are models of $\mathbb{T}^m$ and for all $\vee$-free formulas ξ we have $(D,F) \vDash P_\xi[\mathbf{d}/\mathbf{x}] \Leftrightarrow (D,F) \Vdash \xi[\mathbf{d}/\mathbf{x}]$. Then for any (D,F) in $\mathcal{M}^\subseteq$ and $\mathbf{d} \in D^{l(\mathbf{x})}$ we have: $(D,F) \vDash P_\phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow (D,F) \Vdash \phi[\mathbf{d}/\mathbf{x}] \Rightarrow (D,F) \Vdash (\psi \vee \theta)[\mathbf{d}/\mathbf{x}] \Rightarrow (D,F) \Vdash \psi[\mathbf{d}/\mathbf{x}]$ or $(D,F) \Vdash \theta[\mathbf{d}/\mathbf{x}] \Leftrightarrow (D,F) \vDash P_\psi[\mathbf{d}/\mathbf{x}]$ or $(D,F) \vDash P_\theta[\mathbf{d}/\mathbf{x}] \Leftrightarrow (D,F) \vDash (P_\psi \vee P_\theta)[\mathbf{d}/\mathbf{x}]$. Thus by Lemma 2.2.6, $\mathbb{T}^m$ proves the sequent $P_\phi \vdash_{\mathbf{x}} P_\psi \vee P_\theta$. Therefore, $\mathbb{T}^m$ proves the sequent $P_\phi \vdash_{\mathbf{x}} P_\psi$ or the sequent $P_\phi \vdash_{\mathbf{x}} P_\theta$, whence $\mathbb{T}$ proves $\phi \vdash_{\mathbf{x}} \psi$ or $\phi \vdash_{\mathbf{x}} \theta$.

Finally, for full first-order logic we have:

Corollary 4.1.13 *Let $\mathbb{T}$ be a first-order theory. Then $\mathbb{T}$ has a conservative generalized fallible Beth* model.*

Proof Let $\mathbb{T}^m$ be the regular Morleyization of $\mathbb{T}$ over extended signature Σ^m, with Σ the signature of $\mathbb{T}$. By Theorem 2.2.4 there exists a chase-complete category $\mathcal{M}$ of model diagrams for $\mathbb{T}^m$. By Theorem 4.1.4 the functor $a \circ \mathrm{Ev} : \mathcal{C}_\mathbb{T} \simeq \mathcal{C}_{\mathbb{T}^m} \longrightarrow \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, C)$ is conservative and Heyting. From Lemma 4.1.6, by [11, C2.3.18–19(i)] and [11, C3.1.23], the (right) top functor of the following commutative (up to isomorphism) diagram

$$\begin{tikzcd}
 & \mathrm{Sh}(\mathcal{M}^{\mathrm{op}}, C) \arrow[r, "a\circ\pi^*\circ i"] & \mathrm{Sh}\big(\mathcal{M}^{\subseteq\mathrm{op}}, C\big) \\
\mathcal{C}_\mathbb{T} \simeq \mathcal{C}_{\mathbb{T}^m} \arrow[ur, "a\circ\mathrm{Ev}"] \arrow[r, "\mathrm{Ev}"'] & \mathbf{Set}^{\mathcal{M}} \arrow[u, "a"] \arrow[r, "\pi^*"'] & \mathbf{Set}^{\mathcal{M}^\subseteq} \arrow[u, "a"']
\end{tikzcd}$$

⊣

is Heyting and conservative. Hence so is the composite top functor. As in the proof of 4.1.4, the subpresheaves of the form $\pi^* \circ \mathrm{Ev}_{[\mathbf{x} \mid \phi]} \hookrightarrow \pi^* \circ \mathrm{Ev}_{[\mathbf{x} \mid \top]}$ are C-closed.

Define a generalized fallible Beth structure B as follows. Let P be $\mathcal{M}^\subseteq$, and let the functor $\mathfrak{D} : \mathcal{M}^\subseteq \longrightarrow \mathscr{F}$ send a Σ^m-diagram S to its Σ reduct extended with $[\![\bot]\!]^S := [\![P_\bot]\!]^S$. For $S \in \mathcal{M}^\subseteq$ let $T(S)$ be the set of finite binary trees with nodes in $\mathcal{M}^\subseteq$, root S, and such that the children of any node S' form a chase pair (as given in the paragraph following 4.1.2) over S'.

Let $\mathbf{x}.\phi$ be first-order over Σ, $S \in \mathcal{M}^\subseteq$, and $\mathbf{d}$ a list of the same length as $\mathbf{x}$ of elements in the domain of S . We show by induction on ϕ that $S \Vdash^B \phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow S \vDash P_\phi[\mathbf{d}/\mathbf{x}]$. And, simultaneously, that B satisfies the conditions for being a Beth* structure.

Let ϕ be atomic or $\phi = \bot$. Suppose $S \Vdash^B \phi[\mathbf{d}/\mathbf{x}]$. Then there exists a tree $t \in T(S)$ such that for all leaves S' it is the case that $\mathfrak{D}(S') \vDash \phi[\mathbf{d}]$. Hence $S' \vDash P_\phi[\mathbf{d}/\mathbf{x}]$. Now, the inclusions $S \subseteq S'$ define a C-cover, so $S \vDash P_\phi[\mathbf{d}/\mathbf{x}]$, and thus $\mathfrak{D}(S) \vDash \phi[\mathbf{d}/\mathbf{x}]$. The converse is immediate. The case for the existential quantifier is similar, and the the case for conjuction is immediate.

Let $\phi = \psi \vee \theta$. Suppose $S \Vdash^B \phi[\mathbf{d}/\mathbf{x}]$. Then there exists a tree $t \in T(S)$ such that for all leaves S' it is the case that $S' \Vdash^B \psi[\mathbf{d}]$ or $S' \Vdash^B \psi[\mathbf{d}]$. By induction hypothesis $S' \vDash P_\psi[\mathbf{d}]$ or $S' \vDash P_\theta[\mathbf{d}]$, so $S' \vDash P_\phi[\mathbf{d}]$, and since t defines a C-cover on S, $S \vDash P_\phi[\mathbf{d}]$. Conversely, suppose $S \vDash P_\phi[\mathbf{d}/\mathbf{x}]$. Then $\langle[\mathbf{x} \mid \phi], \mathbf{d}\rangle$ defines a chase pair on S, yielding a tree in $T(S)$ with two leaves S' and S'' such that, by the induction hypothesis, $S' \Vdash^B \psi[\mathbf{d}]$ and $S'' \Vdash^B \phi[\mathbf{d}]$.

Let $\phi = \forall y.\, \psi$. Suppose $S \Vdash^B \phi[\mathbf{d}/\mathbf{x}]$. Then for all $S \subseteq S'$ we have $\mathfrak{D}(S') \Vdash^B \psi[\mathbf{d}/\mathbf{x}, c/y]$ for all c in the domain of S', thus by induction hypothesis $S' \vDash P_\psi[\mathbf{d}, c]$. Hence, since $\pi^* \circ \mathrm{Ev}: \mathcal{C}_\mathbb{T} \simeq \mathcal{C}_{\mathbb{T}^m} \longrightarrow \mathbf{Set}^{\mathcal{M}^\subseteq}$ is Heyting, $S \vDash P_\phi[\mathbf{d}/\mathbf{x}]$. The converse follows from $P_{\forall y.\psi} \vdash^{\mathbb{T}^m}_{\mathbf{x},y} P_\psi$. And $\rightarrow$ is similar.

4.1.14 Beth completeness

For enumerable first-order $\mathbb{T}$ we specialize 4.1.4/4.1.13 to the effect that for every $\mathbb{T}^m$-model diagram (D, F) in suitable $\mathcal{M}$ there is a Beth model $\mathbf{B}$ with root domain D such that $\mathbf{B} \Vdash \phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow (D, F) \vDash P_\phi[\mathbf{d}/\mathbf{x}]$ for all first-order ϕ. This yields a Beth* completeness theorem for $\mathbb{T}$.

Specifically, let $\mathbb{T}$ be a enumerable first-order theory over a signature Σ and $\mathbb{T}^m$ its regular Morleyization. In this section, we assume that the sequent $\top \vdash \exists x.\, x = x$ is an axiom of $\mathbb{T}$. We refer to theories having this sequent as an axiom as *habitative* Let $\mathcal{M}$ be the subcategory of $\mathrm{MDiag}_b(\mathbb{T}^m)$ consisting of diagrams (D, F) of the following form. The domain D is a semi-decidable subset of $\mathbb{N}$, coding a bounded relation from $\mathbb{N}$ to $\mathbb{N}$. D comes equipped with the least upper bound. Denote by f_D the function $\mathrm{f_D} : \mathbb{N} \longrightarrow 2^\mathbb{N}$ such that $D = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}.\, f(n)(m) = 1\}$. The set of facts F is, similarly, (coded as) a semi-decidable subset of $\mathbb{N}$ (with function f_F). Then, straightforwardly, $\mathcal{M}$ is chase-complete (for $\mathbb{T}^m$) with a weakly reflective chase functor. We can assume that if (D, F) is a diagram and $(D', F') = \mathrm{Ch}(D, F)$ then $f_D \leq f_{D'}$ and $f_F \leq f_{F'}$. We fix an enumeration g of (codes of) all possible facts of the form $\langle[\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle$, where every such fact is revisited an infinite number of times.

For any $(D, F) \in \mathcal{M}$ we define a binary tree $\mathcal{T}_{D,F}$ over (D, F), where $\mathcal{T}_{D,F}$ occurs as a subcategory of $\mathcal{M}$, as follows. The diagram (D, F) is the root. At node (D', F') at level n, if $g(n)$ is, say, $\langle[\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle$ and $\exists m \leq n.\, f_{F'}(g(n), m) = 1$ then (D', F') has the children $\mathrm{Ch}(D', F' \cup \{P_\phi[\mathbf{d}]\})$ and $\mathrm{Ch}(D', F' \cup \{P_\psi[\mathbf{d}]\})$. Else the children of (D', F') are both (D', F') itself. In the former case, we say that $\langle[\mathbf{x} \mid P_{\phi\vee\psi}], \mathbf{d}\rangle$ *is chased.*

$\mathcal{T}_{D,F}$ becomes a fallible Beth structure for Σ by setting $P = \mathcal{T}_{D,F}$, and letting the functor $\mathfrak{D} : \mathcal{T}_{D,F} \longrightarrow \mathscr{F}$ send a Σ^m-diagram S to its Σ reduct extended with $[\![\bot]\!]^S := [\![P_\bot]\!]^S$. It is clear that for any node q of $\mathcal{T}_{D,F}$, any level of $\mathcal{T}_{D,F}$ above it can be seen as a $\overline{C}$-cover of q (and therefore also a $\overline{B}$-cover, since $\overline{B} = \overline{C}$ in this setting). We refer to it therefore also as "a cover of q". By the proof of Theorem 4.1.4, we have, therefore, that for any atomic formula $\mathbf{x}.P_\phi$ of Σ^m, any node q in $\mathcal{T}_{D,F}$, and any elements $\mathbf{d}$ of q, if $P_\phi[\mathbf{d}/\mathbf{x}]$ is true on a cover of q then $q \vDash P_\phi[\mathbf{d}/\mathbf{x}]$.

Lemma 4.1.15 *Let $(D,F) \in \mathcal{M}$ and consider the Beth-structure $\mathcal{T}_{D,F}$. Let p be a node in $\mathcal{T}_{D,F}$, let $\mathbf{x}.\phi$ be a first-order formula over Σ, with P_ϕ the corresponding atomic formula in Σ^m, and let $\mathbf{d} \in p$. Then*

$$p \Vdash \phi[\mathbf{d}/\mathbf{x}] \Leftrightarrow p \vDash P_\phi[\mathbf{d}/\mathbf{x}]$$

PROOF By induction on ϕ, as follows.

ϕ atomic or $\phi = \bot$: by the remark immediately preceding this lemma.

$\phi \equiv \varphi \wedge \psi$ or $\phi = \top$: immediate.

$\phi = \varphi \vee \psi$: Assume $p \Vdash (\varphi \vee \psi)[\mathbf{d}]$. Then there exists a cover of p such that for all q in the cover either $q \Vdash \varphi[\mathbf{d}]$ or $q \Vdash \psi[\mathbf{d}]$. By induction hypothesis, then, either $q \vDash P_\varphi[\mathbf{d}]$ or $q \vDash P_\psi[\mathbf{d}]$. As $\mathbb{T}^m$ proves e.g. $P_\varphi \vdash_{\mathbf{x}} P_{\varphi\vee\psi}$ therefore $q \vDash P_{\varphi\vee\psi}[\mathbf{d}]$. Whence $p \vDash P_{\varphi\vee\psi}[\mathbf{d}]$.
Conversely, assume that $p \vDash P_{\varphi\vee\psi}[\mathbf{d}]$. Then for all $q \geq p$, we have $q \vDash P_{\varphi\vee\psi}[\mathbf{d}]$. Therefore, there exists a cover of p, say at level n, such that on that cover $P_{\varphi\vee\psi}[\mathbf{d}]$ is chased. Then for all $q \geq p$ at level $n+1$ we have that either $q \vDash P_\varphi[\mathbf{d}]$ or $q \vDash P_\psi[\mathbf{d}]$. Whence, by induction hypothesis, either $q \Vdash \varphi[\mathbf{d}]$ or $q \Vdash \psi[\mathbf{d}]$, and so $p \Vdash (\varphi \vee \psi)[\mathbf{d}]$.

$\phi = \exists y.\,\varphi$: We have that $p \Vdash \exists y.\,\varphi[\mathbf{d}]$ iff $q \Vdash \varphi[a_q, \mathbf{d}]$ on a cover iff $q \vDash P_\varphi[a_q, \mathbf{d}]$ on a cover iff $q \vDash \exists y.\,P_\varphi[\mathbf{d}]$ on a cover iff $q \vDash P_{\exists x.\varphi}[\mathbf{d}]$ on a cover iff $p \vDash P_{\exists y.\varphi}[\mathbf{d}]$.

$\phi = \varphi \to \psi$: Assume that $p \vDash P_{\varphi\to\psi}[\mathbf{d}]$. Let $q \geq p$ and assume that $q \Vdash \varphi[\mathbf{d}]$. By induction hypothesis $q \vDash P_\varphi[\mathbf{d}]$. Now, $\mathbb{T}^m$ proves the sequent $P_\varphi \wedge P_{\varphi\to\psi} \vdash_{\mathbf{x}} P_\psi$. Whence $q \Vdash P_\psi[\mathbf{d}]$. And so applying the induction hypothesis again, $q \Vdash \psi[\mathbf{d}]$. Hence $p \Vdash (\varphi \to \psi)[\mathbf{d}]$.
For the converse direction, observe first that for $(D,F) \in \mathcal{M}$ we have that $(D,F) \vDash P_{\varphi\to\psi}[\mathbf{d}]$ iff $\mathrm{Ch}(D, F \cup \{P_\varphi[\mathbf{d}]\}) \vDash P_\psi[\mathbf{d}]$: for the right-to-left direction, the right hand side implies that there is a $\mathbb{T}^m$-provable sequent $P_\chi \wedge P_\phi \vdash_{\mathbf{x}} P_\psi$ such that $(D,F) \vDash P_\chi[\mathbf{d}/\mathbf{x}]$. Whence $\mathbb{T}$ proves $\chi \wedge \phi \vdash_{\mathbf{x}} \psi$ and therefore $\chi \vdash_{\mathbf{x}} \phi \to \psi$, so that $\mathbb{T}^m$ proves $P_\chi \vdash_{\mathbf{x}} P_{\phi\to\psi}$. Now, assume that $p \Vdash (\varphi \to \psi)[\mathbf{d}]$. We have $p \vDash P_{\top\vee\varphi}[\mathbf{d}]$, so there exists a level n where it is chased. For a node $q = (D', F')$ on level n, the right child is therefore $q' = \mathrm{Ch}(D, F \cup \{P_\varphi(\mathbf{d})\})$. By induction hypothesis, $q' \vDash P_\varphi[\mathbf{d}]$ implies that $q' \Vdash \varphi[\mathbf{d}]$. Thus since $q' \geq p$ we have by assumption that $q' \Vdash \psi[\mathbf{d}]$, so $q' \vDash P_\psi[\mathbf{d}]$. So, by the observation, $q \vDash P_{\varphi\to\psi}[\mathbf{d}]$. With $P_{\varphi\to\psi}[\mathbf{d}]$ true on a cover of p we have, then, that $p \vDash P_{\varphi\to\psi}[\mathbf{d}]$.

$\phi = \forall y. \varphi$: Assume that $p \vDash P_{\forall y.\varphi}[\mathbf{d}]$. Then for all $q \geq p$ and $a \in q$ we have that $q \vDash P_{\varphi}[a, \mathbf{d}]$, by applying the axiom $P_{\forall y.\varphi} \vdash_{y,\mathbf{x}} P_{\varphi}$ of $\mathbb{T}^m$. So, by induction hypothesis, $q \Vdash \varphi[a, \mathbf{d}]$. Thus $p \Vdash \forall x. \varphi[\mathbf{d}]$.
For the converse direction, observe first, similar to the case of the conditional above, that that for $(D, F) \in \mathcal{M}$ we have that $(D, F) \vDash P_{\forall x.\varphi}[\mathbf{d}]$ iff $\mathrm{Ch}(D + 1, F) \vDash P_{\varphi}[e, \mathbf{d}]$, for all elements e of $\mathrm{Ch}(D + 1, F)$ (specifically the new element of $D + 1$). Since $\mathbb{T}$ is habitative, this implies that $(D, F) \vDash P_{\forall x.\varphi}[\mathbf{d}]$ iff $\mathrm{Ch}(D, F) \vDash P_{\varphi}[e, \mathbf{d}]$, for all elements e of $\mathrm{Ch}(D, F)$ (specifically any fresh element e added by an application of the axiom $\top \vdash \exists x. x = x$). Assume, then, that $p \Vdash \forall x. \varphi[\mathbf{d}]$. We have $p \vDash P_{\top\vee\top}$, so there exist a level n on which $P_{\top\vee\top}$ is chased. For $(D', F') = q \geq p$ on level n, either child is $\mathrm{Ch}(D', F')$ and $\mathrm{Ch}(D', F') \Vdash \varphi[e, \mathbf{d}]$, for all elements e in (D', F'). By induction hypothesis, $\mathrm{Ch}(D', F') \vDash P_{\varphi}[e, \mathbf{d}]$, so $q = (D', F') \vDash P_{\forall x.\varphi}[\mathbf{d}]$. With $P_{\forall x.\varphi}[\mathbf{d}]$ thus true on a cover, we conclude $p \vDash P_{\forall x.\varphi}[\mathbf{d}]$. $\dashv$

It is clear that given a Beth* model of a first order $\mathbb{T}$, the nodes, and the root in particular, model $\mathbb{T}^m$. We can now state the converse

Theorem 4.1.16 *Let $\mathbb{T}$ be a habitative enumerable first-order theory. Let $\mathbb{T}^m$ be its regular Morleyization. For every enumerable Tarski model $\mathbf{M}$ of $\mathbb{T}^m$ there exists a fallible Beth* model $\mathbf{B}$ of $\mathbb{T}$ such that the domain of the root r is the domain of $\mathbf{M}$, and such that for all $\mathbf{m} \in \mathbf{M}^{l(\mathbf{m})}$ and $\mathbf{x}.\phi$ first-order,*

$$r \Vdash \phi[\mathbf{m}/\mathbf{x}] \Leftrightarrow \mathbf{M} \vDash P_{\phi}[\mathbf{m}/\mathbf{x}]$$

The following can then be seen as a constructive version of the completeness theorem of [10].

Corollary 4.1.17 *Let $\mathbb{T}$ be a habitative enumerable first-order theory. Then $\mathbb{T}$ is complete with respect to fallible Beth* models.*

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References

[1] Serge Abiteboul, Richard Hull, and Victor Vianu. *Foundations of Databases*. Addison-Wesley, 1995.

[2] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.

[3] J. L. Bell and M. Machover. *A course in mathematical logic*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[4] Stefano Berardi and Silvio Valentini. Krivine's intuitionistic proof of classical completeness (for countable languages). *Annals of Pure and Applied Logic*, 129:93–106, 2004.

[5] Carsten Butz. Saturated models of intuitionistic theories. *Annals of Pure and Applied Logic*, 129:245–275, 2002.

[6] Thierry Coquand and Jan Smith. An application of constructive completeness. In *Proceedings of the Workshop TYPES'95, Torino, Italy, June 1995*, pages 76–84, 1995.

[7] Michel Coste, Henri Lombardi, and Marie-Françoise Roy. Dynamical methods in algebra: Effective Nullstellensätze. *Annals of Pure and Applied Logic*, 3:203–256, 2001.

[8] Harrie de Swart. Another intuitionistic completeness proof. *Journal of Symbolic Logic*, 41(3):644–662, 1976.

[9] Henrik Forssell and Peter Lefanu Lumsdaine. Logic for non-discrete signatures. Will be available at `http://arxiv.org/find/math/1/au:+Forssell_H/0/1/0/all/0/1`.

[10] Dov M. Gabbay. A new version of Beth semantics for intuitionistic logic. *Journal of Symbolic Logic*, 42(2):306–308, 1977.

[11] Peter T. Johnstone. *Sketches of an Elephant*, volume 43 and 44 of *Oxford Logic Guides*. Clarendon Press, Oxford, 2002.

[12] Georg Kreisel. On weak completeness of intuitionistic predicate logic. *The Journal of Symbolic Logic*, 27(Special issue 2):139–158, 1962.

[13] Jean-Louis Krivine. Une preuve formelle et intuitionistique du théoreme de completude de la logique classique. *The Bulletin of Symbolic Logic*, 2(4):405–421, 1996.

[14] Iris Loeb. Equivalents of the (Weak) Fan Theorem. *Annals of Pure and applied Logic*, 132:51–66, 2005.

[15] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. Springer–Verlag, New York, 1992.

[16] Michael Makkai. A theorem on Barr-exact categories, with an infinitary generalization. *Annals of Pure and Applied Logic*, 47:225–268, 1990.

[17] Michael Makkai. On Gabbay's proof of the Craig interpolation theorem for intuitionistic predicate logic. *Notre Dame Journal of Formal Logic*, 36(3):364–381, 1995.

[18] Michael Makkai and Gonzalo E. Reyes. *First Order Categorical Logic.* Number 611 in Lecture Notes in Mathematics. Springer, 1977.

[19] Charles McCarty. Completeness and incompleteness for intuitionistic logic. *Journal of Symbolic Logic*, 73(4):1315–1327, 2008.

[20] Charles McCarty. On theorems of Gödel and Kreisel: Completeness and Markov's Principle. *Notre Dame Journal of Formal Logic*, 35(1):?, 2008.

[21] Ieke Moerdijk and Erik Palmgren. Type theories, toposes and constructive set theory: Predicative aspects of AST. *Annals of Pure and Applied Logic*, 114:155–201, 2002.

[22] Sara Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Archive of Mathematical Logic*, 42(4):389–401, 2003.

[23] Anne S. Troelstra. *Mathematical Investigation of Intuitionistic arithmetic and Analysis.* Springer, 1973.

[24] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 1996.

[25] Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics*, volume 1. North Holland, 1988.

[26] Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics*, volume 2. Elsevier, 1988.

[27] Wim Veldman. An intuitionistic completeness theorem for intuitionistic predicate logic. *Journal of Symbolic Logic*, 41(1):159–166, 1976.