ENUMERATIVE COMBINATORICS AND SYMMETRIC FUNCTIONS

Supervisor: Per Alexandersson, per.alexandersson@math.su.se.

1. Overview

My research interests mainly concerns combinatorial objects and associated combinatorially defined polynomials (or symmetric functions). For example, permutations, lattice paths, Young tableaux, set partitions, rook placements, graphs, graph colorings, posets, polytopes and matroids. The properties of polynomials I am interested in are real-rootedness (see [Brä15]), unimodality, recursions, enumerative properties, values at roots of unity (cyclic sieving, see [RSW04]). For polynomials in several variables, one can ask questions about stability, the Lorentzian property or M-convexity.

For symmetric functions, the main question is usually Schur positivity, where combinatorial methods such as RSK or crystal graphs are used. See www.symmetricfunctions.com for a survey of topics.

The projects listed below heavily involves computer experimentation (Mathematica, Sage, Python or Rust), so the applicant should expect to do lot of programming.

2. Suggested projects

- Study the recently introduced class of *rook matroids*, [AJ24]. There are several open questions left in this area, in particular in connection with polytopes. Perhaps one can leverage matroid theory to resolve more cases of the log-concavity part of the Negger–Stanley conjecture.
- Prove that certain generalizations of Schur functions (arising from certain Young tableaux) are Schur-positive. This generalization is motivated by its connection to an old conjecture of King–Tollu–Toumazet [KTT04]. Most likely, there is also a representation-theoretical aspect explaining the Schur-positivity. We already have a conjectured combinatorial interpretation of the coefficients in the Schur expansion.
- Prove real-rootedness of the sink-counting polynomial associated with acyclic orientations of claw-free graphs. The class of claw-free graphs show up in several theorems and conjectures regarding polynomials with real roots. I conjecture that if one considers such a graph, and sum over all its acyclic orientations θ and count the number of sinks (vertices with only in-directed edges) then the polynomial $\sum_\theta t^{sinks(\theta)}$ has only real roots.
- Explore cyclic unit-interval graphs and its associated chromatic symmetric functions and its associated LLT polynomials. Many conjectures are still open about these (Schur positivity in particular).

References

[AJ24] Per Alexandersson and Aryaman Jal, *Rook matroids and log-concavity of p-Eulerian polynomials*.

[Brä15] Petter Brändén, *Unimodality, log-concavity, real–rootedness and beyond*, Handbook of Enumerative Combinatorics, Chapman and Hall/CRC, March 2015, pp. 437–483.

[KTT04] R. C. King, C. Tollu, and F. Toumazet, *Stretched Littlewood–Richardson coefficients and Kostka coefficients*, Symmetry in Physics: In Memory of Robert T. Sharp (P. Winternitz, J. Harnard, C. S. Lam, and J. Patera, eds.), vol. 34, AMS / OUP, 2004, pp. 99–112.

[RSW04] Victor Reiner, Dennis Stanton, and Dennis E. White, *The cyclic sieving phenomenon*, Journal of Combinatorial Theory, Series A **108** (2004), no. 1, 17–50.