TOPICS IN COMPLEX ANALYSIS, HARMONIC ANALYSIS, AND COMPLEX GEOMETRY

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Overview

Generally speaking, I have broad interests in mathematical analysis, including **complex analysis (in one and in several variables); harmonic analysis; functional analysis; probability theory and stochastic processes; and dynamical systems**, and I would be happy to advise a PhD student who wishes to work in one or more of these subfields.

Here is a selection of some specific topics I have worked on in the last few years, with salient references listed.

- **Stable polynomials and boundary behavior of rational functions in $\mathbb{C}^d$**:

  A polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ is said to be stable with respect to some open set $\Omega \subset \mathbb{C}^d$ if p has no zeros in Ω; p may well have zeros in the closure of Ω. Such polynomials arise naturally in several contexts; for instance, they are numerators of rational functions that are holomorphic in Ω. How can boundary properties (integrability, non-tangential regularity) of such q/p be related to the geometry of how the zero set of p approaches $\partial\Omega$? Does the requirement that p be non-vanishing force p to have special algebraic properties, and can this be used to further probe rational functions having p as their denominators? We have a reasonably good understanding what the answer to these questions is in the case of the two-dimensional polydisk, and the natural progression would likely be to examine (a) other two-dimensional domains, and (b) higher dimensions.

  - (with K. Bickel and J.E. Pascoe) *Singularities of rational inner functions in higher dimensions*, Amer. J. Math. 144 (2022).
  - (with K. Bickel, G. Knese, and J.E. Pascoe) *Stable polynomials and admissible numerators in product domains*, Bull. London Math. Soc. 57 (2025).

- **Clark theory in several variables**:

  A bounded holomorphic function is called inner in $\Omega \subset \mathbb{C}^d$ if its boundary values (in a suitable sense) are unimodular. Given an inner function ϕ, we consider the expression

$$\frac{1-|\phi(z)|^2}{|\alpha-\phi(z)|^2}, \quad z \in \Omega,$$

and note that this is a positive pluriharmonic function. As such, it can (for reasonable Ω) be represented as the Poisson integral of a special type of positive measure μ_ϕ which is supported on part of, or all of, $\partial\Omega$. How does the structure of this measure reflect the nature of the inner function ϕ? How do operations of ϕ translate to modifications of μ_ϕ? What can we say about the structure of the space $L^2(\mu_\sigma)$. These questions have been addressed in some detail for the subclass of rational inner functions, including in a very nice recent paper by Calzi, but detailed results are lacking for more general classes of inner functions.

Clark measures are of considerable interest in operator theory because of the existence of a unitary operator $J\colon (\phi H^2)^\perp \to L^2(\mu_\sigma)$ that intertwines the compressed shift on the so-called model space $(\phi H^2)^\perp$ and multiplication by z on the L^2 space of the Clark measure. Some initial extensions in the setting of higher dimensions have been established (eg. the analog of the Clark operator is isometric but not always a unitary), but much certainly remains to be done to relate multi-variable operator theory to multiplication operators on $L^2(\sigma_\phi)$ when ϕ is a multivariable inner function.

- (with K. Bickel and J.A. Cima) *Clark measures for rational inner functions*, Michigan Math. J. 73 (2023).
- (with J.T. Anderson, L. Bergvist, K. Bickel, and J.A. Cima) *Clark measures for rational inner functions II*, Ark. Mat. 62 (2024).
- M. Calzi, *Clark measures associated with rational inner functions on bounded symmetric domains*, Proc. AMS, to appear (2025).

• **Multipliers on Hardy-Orlicz and other function spaces**:

Suppose $\mathcal{X}$ is a function space on $\mathbb{T}$, the circle, or on $\mathbb{T}^d$, which is defined in terms of properties of Fourier or Taylor coefficients $\{\hat{f}_n\}$. A sequence $\{\lambda_n\}$ is said to be a multiplier from $\mathcal{X}$ into ℓ^p if we have $\|\{\hat{f}_n\lambda_n\}\|_{\ell^p} \leq C\|f\|_{\mathcal{X}}$ for the respective norms. Given a specific space $\mathcal{X}$ of interest, can one characterize multipliers in a concrete and useful way? Using such characterizations, is it possible to devise simple necessary conditions (decay, regularity, etc) that f needs to satisfy in order to belong to $\mathcal{X}$, whose initial norm need not be defined in a straight-forward manner?

With several collaborators, including my SU colleague Salvador Rodríguez-López, we have been studying such questions for Hardy-Orlicz spaces. One example of such a space is $H^{\log}(\mathbb{T})$ which informally speaking consists of functions whose maximal functions $M[f]$ have

$$\int_{\mathbb{T}} \Psi_0(M[f](x))dx < \infty$$

(with appropriate adjustments to capture distributions with this type of property), where $\Psi_0(t) = t/\log(e+t)$. In this case, one result we have obtained is that $f \in H^{\log}$

implies that

$$\sum_{n\in\mathbb{Z}} \frac{|\hat{f}_n|}{(|n|+1)\log(|n|+1)} < \infty.$$

A simple sample application of this type of result is that the Dirac delta does not belong to $H^{\log}$ (as can be checked in other ways).

– (with O. Bakas, S. Rodríguez-López, and S. Pott) *Multipliers for Hardy-Orlicz spaces and applications*, Journal d'Analyse Mathématique, to appear (2025).

Prerequisites

I would expect a student who wishes to work with me to have a strong background in pure mathematics in general, and in complex/harmonic/functional analysis in particular. In addition, I think it is important for students to be willing to learn about other fields in mathematics, including (but not limited to) algebraic geometry, differential geometry, dynamical systems, operator theory, etc.

Projects

In my view, it is best to try to arrive at a first project that suits the strengths and interests of the particular student, as opposed to having a fixed direction in mind from the outset. The above can be seen as some indication of my recent interests and directions where I could likely propose concrete problems at a reasonable level, but I am quite open-minded regarding suggestions from the potential student.

I strongly believe that patience, persistence, and an interest in lucid exposition are crucial in order to achieve success as a mathematician, and are in many ways more important than flashy brilliance and speed.

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