Discrete probability theory

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Discrete probability theory is a vast area of research covering objects such as random walks, random graphs, percolation processes, and much more. Such random processes arise naturally when pursuing a mathematical explanation of phenomena that can be observed in the world around us. A large probabilistic toolbox has been developed in order to capture the behaviour of such processes, including Markov chains, martingale methods and coupling techniques. The insights these brings contributes to the understanding in other fields, from physics to data science.

The field is vast enough to accommodate a range of different preferences and skill sets. A suitable topic for a PhD project could be elaborated upon together with the applicant to fit his/her taste and background. Below I outline a specific suggestion in some detail, to give a flavour of what a research problem in the area could look like.

Topic suggestion: Annihilating branching random walks

Random processes with reinforcement are stochastic processes that exhibit long-term coordination, in the sense that the entire trajectory of the process contributes to its eventual fate. Polya's urn model is the standard example of a process of this type. In this model, a finite number of red and blue balls are initially placed in an urn. In each round of the process, a ball is drawn uniformly at random from the urn, and replaced together with a ball of the same colour. The drawn colour is thus reinforced, in that balls of the same colour are more likely to be drawn in future steps. The effect of the reinforcement declines over time, resulting in the early steps of the process having a lasting effect over its continued evolution, and the limiting proportion of red balls in Pólya's urn being random.

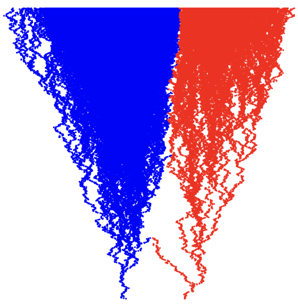

Figure 1: The formation of a linear interface between red and blue in annihilating branching Brownian motion. The slope of the interface is random, as a consequence of the reinforcement mechanism built into the process.

Branching processes, as well as other martingale methods, are central techniques in the study of random processes with reinforcement. *Branching random walks* (and *branching Brownian motion*) are examples of reinforcement processes, particles move around according to random walks (or Brownian motion), and split into independent particles at unit rate. In *annihilating* versions of such processes, particles may be of one out of two types (red and blue, say), and particles of different type annihilate upon contact. The annihilating behaviour turns the growth process into a model for competing growth.

Annihilating branching random walks have in the past decade been found to behave differently depending of the graph they evolve on. On finite graphs coexistence of the two types is not possible [2], whereas in an infinite setting it is [1]. However, much remains unknown regarding these processes, and several open problems and conjectures are stated in the mentioned papers. For instance, in the unbounded setting, on the event of coexistence, the interface separating red and blue particles is known to evolve linearly. The slope of the interface is random, as a consequence of the reinforcement mechanism (see figure). However, finer details regarding the slope of the interface remain unknown, such as its fluctuations around its random limit. These open problems could be the tentative topic of a PhD project.

Animated introductions to some of the results described above:
Annihilating branching Brownian motion
Multi-colour competition with reinforcement

References

[1] Daniel Ahlberg, Omer Angel, and Brett Kolesnik. Annihilating branching Brownian motion. *Int. Math. Res. Not. IMRN*, (13):10425–10448, 2024.

[2] Daniel Ahlberg, Simon Griffiths, Svante Janson, and Robert Morris. Competition in growth and urns. *Random Structures Algorithms*, 54(2):211–227, 2019.