Topics in additive combinatorics: number theory, harmonic analysis and probability

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Questions about set addition have been studied for centuries, particularly in number theory. We know, for instance, that any positive integer can be written as the sum of four squares, and that every odd integer greater than 5 can be written as a sum of three primes. However, even about these specific sets of integers — squares and primes — there are many things we do not know. A relatively recent trend ('additive combinatorics') is to try to understand set addition much more generally, and in particular to understand under what conditions on a set A of integers we can deduce interesting additive information. In fact, it is often fruitful to consider subsets not just of the integers, but of abelian groups more generally.

My research is largely concerned with improving our understanding of additive structures in subsets A of abelian groups, and their sumsets $A + A$, $A + A + A$ etc (and to some extent their non-abelian analogues). In particular, what I find the most interesting is the application (and development) of techniques in fields such as harmonic analysis and probability to make advances in the area.

Specific projects will be highly student-dependent. One possibility is to investigate generalisations and limitations of recent advances in the study of three-term arithmetic progressions in sets of integers. It has been known since work of van der Corput (1939) that the primes contain infinitely many three-term arithmetic progressions. More recently, there has been substantial progress [1, 7, 2] on quantitative versions of Roth's theorem, notably the breakthrough of Kelley–Meka, which imply the result about primes simply by virtue of there being 'many' primes in intervals $\{1, 2, \ldots, N\}$. The techniques use (among other things) harmonic-analytic almost-periodicity ideas (with roots in [4]), as well as averaging arguments that are naturally viewed probabilistically. In parallel, algebraic methods [3, 5] (e.g. vector spaces of polynomials over finite fields) and information-theoretic/entropy ideas [6] have led to striking progress on related problems. A possible direction for a PhD project is to understand how these approaches might be unified or extended, and to clarify their inherent limitations.

Background in additive combinatorics, combinatorics, analytic number theory and/or Fourier analysis is helpful; strong applicants may come from any of these backgrounds.

References

[1] T. F. Bloom and O. Sisask, *Breaking the logarithmic barrier in Roth's theorem on arithmetic progressions,* `https://arxiv.org/abs/2007.03528`.

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[3] E. Croot, V. F. Lev and P. Pach, *Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small,* Ann. of Math. 185 (2017), 331–337.

[4] E. Croot and O. Sisask, *A probabilistic technique for finding almost-periods of convolutions,* Geom. Funct. Anal. 20 (2010), no. 6, 1367–1396.

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[6] W. T. Gowers, B. Green, F. Manners, T. Tao, *On a conjecture of Marton,* Ann. of Math. (2) 201 (2025), no. 2, 515–549.

[7] Z. Kelley and R. Meka, *Strong Bounds for 3-Progressions,* `https://arxiv.org/abs/2302.05537`.