Recursive U-quantiles

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Recursive U-quantiles

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Abstract

Suppose we have a function \( h \) with a sequence of i.i.d. random variables \( \{X_i\}_{i=1}^{\infty} \) with marginal distribution \( F \). Let \( H_r \) be the distribution of \( h(X_1, \ldots, X_r) \), \( r \geq 2 \). We consider on-line schemes for estimating quantiles of \( H_r \). Such an estimator is based on a design \( D_m \), which is a small subset of all \( \binom{\mathbb{N}}{m} \) possible index vectors \( I = (i_1, \ldots, i_m) \) having distinct entries not exceeding \( m \). When a new observation \( X_{m+1} \) arrives, \( \gamma = (D_m \setminus D_{m-1}) \) new vectors \( (X_{m+1}, \ldots, X_{m+r}) \) with \( I \in D_m \setminus D_{m-1} \) are used to modify the current estimate. When \( m \to \infty \), the asymptotic relative efficiency of the recursive estimator compared to the off-line estimator (U-quantile) tends to one. The on-line estimator is closely related to incomplete U-quantiles (Høssjer, 1996), and it generalizes a recursive quantile estimator considered by Holst (1987) for \( m = 1 \).

1 Introduction

Assume we have a sequence \( \{X_i\}_{i=1}^{\infty} \) of \( \mathbb{R} \)-measurable random variables that are independent and identically distributed (i.i.d.) with common distribution \( F \). Let \( h : \mathbb{R}^m \to \mathbb{R} \) be a measurable function, and define another distribution function \( H_r(f) = P(h(X_1, \ldots, X_r) \leq f) \), which depends on \( F \) and \( h \). We consider estimating the quantile

\[ F^{-1}(p) = \inf\{x; F(x) \geq p\}, \]

given some fixed \( 0 < p < 1 \). For each \( I = (i_1, \ldots, i_m) \), introduce the short-hand notation \( h(X_{i_1}, \ldots, X_{i_m}) \). Let also \( S_m(n) = \{I = (i_1, \ldots, i_m); 1 \leq i_1 \leq \ldots \leq i_m \leq n \} \) be the collection of all \( \binom{n}{m} \) possible multi-indices \( I \) with entries not exceeding \( n \). For any design \( D_m \subseteq S_m(n) \) of multindices, we may define the distribution function

\[ R_m(f) = \frac{1}{|D_m|} \sum_{I \in D_m} \mathbf{1}_{h(X_i) \leq f}. \]
which is an empirical analog of $H_T$. Here $N(x)$ is the number of elements contained in $D_n$. A natural estimator of $\phi$ is

$$\hat{\phi} = \frac{N(x)}{n}.$$

If $D_n = \mathcal{S}(m)$, $\hat{\phi}$ is a $U$-quantile (UQ). The most well known UQ is the Hodges-Lehmann estimator, which is the median of all $\frac{1}{2} (x_i + x_j)$ in the location model (Hodges and Lehmann, 1962). The UQ based on the kernel $k(x,y) = \delta_{\epsilon} - \delta_{\epsilon}$ results in a measure of spread, with $\epsilon > 0$ a constant that ensures consistency. If we want to estimate the standard deviation, interquartile range or some other scale functional (cf. Beekel and Lehmann, 1979, Choudhury and Serfling, 1985 and Rousseeuw and Croux, 1993). Another UQ is the Theil-Sen estimator, which is the median of all $\frac{x_i}{x_j}$ in the location model (Hedges, 1981, 1985 and Sen, 1968).

If $D_n \neq \mathcal{S}(m)$, $\hat{\phi}$ is an incomplete $U$-quantile (IUQ). This notion was introduced in Huisjer (1996), but an IUQ estimator was already considered by Brown and Kildea (1978) for the Hodges-Lehmann kernel. By generalizing quantiles to arbitrary L-functionals we obtain so called generalized L-statistics (Serfling, 1984) when $D_n = \mathcal{S}(m)$ and incomplete generalized L-statistics (Huisjer, 1996) for general $D_n$.

There are several advantages of using an incomplete design $D_n$. Since $\frac{N(x)}{n} = O(\omega)$, the computation of $\hat{\phi}$ may be tractable for large $n$ and $m \geq 2$. On the other hand, it is possible to choose designs with $N(x) = O(n)$ and asymptotic relative efficiency (ARE) arbitrarily close to one w.r.t. the corresponding UQ. This phenomenon was first noted by Blom (1976) for incomplete $U$-statistics (defined as $f(x_1,\ldots,x_m)$). Certain IUQ can be used for estimating the scale parameter in nonparametric regression with homoscedastic errors, and they can also be used in time series applications (Huisjer, 1996).

In this paper, we will focus on another application of incomplete designs: On-line estimation of $\phi$. Following Huisjer (1996), we refer to a design as recursive and on-line (RO) if $D_{n+1} \subseteq D_n$ for all $n \geq 2$.

This means that $D_n$ is generated from $D_{n+1}$ by simply adding a number of multi-indices, and this number doesn't increase with $n$. The two designs considered here are (cf. Huisjer, 1996, Section 2)

(D1) RO design based on cyclic permutations: Given a positive integer $y \in \mathbb{Z}^+$, define vectors $k = (i_1, \ldots, i_m)$, $d_k = (d_{k,1}, \ldots, d_{k,m})$ of length $m$, so that all $d_{k,j} \neq d_{k',j}$ for $k \neq k'$ are different, $0 \leq d_{k,j} < d_{k,j+1} < \cdots < d_{k,m}$. Then put $D_n = \{x + d_{k,j}; 1 \leq j \leq y, (i_1, \ldots, i_m) \in \mathcal{D}_n\}$.

Example: $m = 2$, $d_1 = (0, 4)$, $d_2 = (0, 1, 4, 6)$. $n = 4$, $y = 1$ and $d = (0, 1, 4, 6)$.

$D_n = \{x + d_{k,j}; 1 \leq j \leq y, (i_1, \ldots, i_m) \in \mathcal{D}_n\}$ for some $\gamma \in \mathbb{Z}$.

(D2) RO design, $m = 3$, $D_n = \{(i,j); 1 \leq i \leq n, j = 1 \leq \gamma \}$ for some $\gamma \in \mathbb{Z}$.

In fact, both (D1) and (D2) satisfy

$$\{D_n \cup D_{n+1}; \gamma \} = \gamma \{D_n \cup D_{n+1}; 1 \leq \gamma \} \quad \text{for some} \quad \gamma \geq n.$$
with \( m - 1 = \ell \), for (D1) and \( \bar{m} = 1 + \gamma \) for (D2). Hence, the number of added 1's remains fixed for large \( n \). We imposed that all \( \delta \) are different for (D1) to ensure that estimators based on this design have a tractable asymptotic variance. A detailed account of various designs that have been used in the incomplete U-statistics literature may be found in Lee (1990, Chapter 4).

Before introducing our recursive estimator, notice that \( \hat{\theta}_n \) may be written as an M estimator

\[
\sum_{i \neq j} \psi \left( \theta \bar{X}_i - \bar{X}_j \right) = 0.
\]

with score function

\[
\psi(x) = \begin{cases} 
1 & x > 0, \\
1 - p, & x \leq 0.
\end{cases}
\]

To define a recursive estimator of \( \theta \), let \( \hat{\delta}_1 \) and \( \hat{\lambda}_n \) be fixed numbers, and put

\[
\hat{\theta}_n = \hat{\delta} + \frac{1}{n^{\ell p / \ell + 1}} \sum_{i \neq j \neq k} \psi \left( \theta \bar{X}_{ik} - \bar{X}_{ij} \right) - \lambda_n h_n, 
\]

(1.2)

for \( n \geq m \), with

\[
h_n = \frac{1}{[P_n + Q_n]^2}.
\]

Here \( p \geq 0 \) are fixed numbers, \( P_n = \max(0, \min(n, n)) \) and \( h_n \) is a recursive density estimator of \( hF(\theta) \). Finally, \( R \) is a non-negative function that integrates to one and \( r \) a fixed positive number.

If \( m = 1 \) and \( \delta = \delta(1) \), \( \hat{\theta}_n \) is essentially the recursive estimator of \( \theta \) considered by Holst (1987).

In Section 2, we first review some asymptotic theory for (incomplete) U-quantiles and then, in Section 3, we consider the asymptotic behaviour of \( \hat{\theta}_n \). Our main result (Theorem 1) is that \( \hat{\theta}_n \) is asymptotically equivalent to an ITQ based on the same design ((D1) and (D2) respectively). The (ARE) of \( \hat{\theta}_n \) w.r.t. the corresponding U-quantile approaches 1 as \( n \to \infty \). Hence, we have found an on-line estimator of \( \theta \) with negligible loss in asymptotic efficiency. Finally, the proof of Theorem 1 is given Section 4.

2 Asymptotics results for incomplete U-quantiles

Serfling (1984) considered generalized L-statistics (and in particular U-quantiles) as statistical functionals, operating on the U-process \( \bar{X}_n \). This approach was also adopted by Hossjer (1996) for incomplete generalized L-statistics. The linear, first order von Mises expansion of \( \bar{X}_n \) is

\[
\bar{X}_n = \delta + \frac{1}{n^{\ell + 1}} \sum_{i \neq j} A(\bar{X}_i, \bar{X}_j) + \bar{U}_n,
\]

(2.1)

with \( A(\bar{X}, \bar{X}) = \psi(hF(\bar{X})) - \psi(1/2) \). Here \( \bar{U}_n \) is a remainder term of Bahadur type. It has been analyzed by Chandra and Serfling (1995) and Arcones (1995) for U-quantiles. The linear main term in (2.1) is an incomplete U-statistic. Asymptotic normality of \( \bar{X}_n \) is established using asymptotic theory of incomplete U-statistics and proving that \( \bar{U}_n \) is negligible. To this end we need some notation:

Let

\[
\sigma_n^2 = E\left( A(\bar{X}_1, \bar{X}_1) \right).
\]
with \( I \) a cyclic rearrangement of \( \{1, \ldots, m\} \) \( m \) in position \( i \), and \( I_j \) is a cyclic rearrangement of \( \{1, m + 1, \ldots, 2m - 1\} \) with \( 1 \) in position \( j \) Let also
\[
\sigma^2 = \sum_{i=1}^m \sigma^2_i
\]
and
\[
\sigma^2_{m} = (\lambda_{m} + 1)^2
\]
The following result is a special case of Theorem 4.1 in Hüsler (1996):

**Theorem 1** Suppose \( \sigma^2 > 0 \) and that \( HF \) has a positive derivative \( hF(0) \) at \( 0 \). Then, an IUQ based on designs \((D_1)\) or \((D_2)\) has an asymptotically normal distribution,
\[
\sqrt{n}(\hat{D} - \theta) \Rightarrow N(0, \sigma^2(\gamma))
\]
with asymptotic variance given by
\[
\sigma^2(\gamma) = \sigma^2 + \frac{\sigma^2(m - 1)^2}{\gamma}
\]
Notice that \( \sigma^2(\gamma) \rightarrow \sigma^2 \) as \( \gamma \rightarrow -\infty \), which is the asymptotic variance for \( U \)-quantiles (Serfling, 1984). By choosing \( \gamma \) sufficiently large, we obtain an asymptotic relative efficiency arbitrarily close to one.

If \( h \) is symmetric w.r.t. permutation of indices, the asymptotic variance simplifies to
\[
\sigma^2(\gamma) = \sigma^2 + \frac{\sigma^2(m - 1)^2}{\gamma}
\]
with \( \sigma^2 = n^{2p} \left( \sum_{i=1}^{m} d(X_1, X_2, \ldots, X_m) d(X_{m+1}, \ldots, X_m) \right) \).

### 3 On-line estimator

Consider now the recursive estimator \( \hat{D}_r \) defined in Section 2. We will prove below that \( \hat{D}_r \Rightarrow \theta \) and \( \hat{D}_r \Rightarrow hF(0) \). In fact, \( \hat{D}_r \) is a recursive kernel density estimator of \( hF(0) \). Heuristically, this means
\[
\hat{D}_r \Rightarrow hF(0)
\]
In view of (2.1), this motivates why \( \hat{D}_r \) is asymptotically equivalent to an IUQ based on the same recursive design.

We will impose the following regularity conditions:

**A.** \( \hat{D}_r, \ldots, \hat{D}_{m-1}, \hat{D}_{m}, \ldots, \hat{D}_{m+n} \), are arbitrary finite numbers.

**B.** In some neighborhood \( U \) of \( \theta \) and for some \( 0 < \varepsilon_1, \varepsilon_2 < 1 \), \( H_{m} - h_{m} \) exists and is Holder continuous of order \( \varepsilon_1 \) for some \( L > 0 \) and \( m \) such that \( L \geq m \), \( \varepsilon_1 \varepsilon_2 < 1 \), and \( \varepsilon_1 \varepsilon_2 < 1 \) for all \( 2 \leq m \).

**C.** \( H_{m} \) is Holder continuous of order \( \eta \), \( 0 < \varepsilon_1 < 1 \), i.e. \( \left| H_{m}(x) - H_{m}(y) \right| \leq L|x - y|^{s} \) whenever \( x, y \in U \).
(D) For some \( r_1 > 0, 0 \leq r < r_1 \leq 1/2. \)

(E) The kernel function \( K \) satisfies \( \int K(x)dx = 1 \), has compact support, is non-negative, bounded and Lipschitz continuous, i.e. for some \( L < \infty \) we have \( |K(t) - K(s)| \leq Lt - |s| \).

**Theorem 2** Assume a design of type (D1) or (D2), that \( H_F \) has a positive derivative at \( \theta \), and that (A)-(E) hold. Then \( \tilde{\theta} \) has an asymptotically normal distribution,

\[ \sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2(\gamma)) \]

with \( \sigma^2(\gamma) \) as defined in Theorem 1.

**4 Proof of Theorem 2**

Throughout this section, \( C \) will refer to a constant whose value may change from line to line. Unless otherwise stated all convergence means \( \mathbb{P} \), i.e. convergence almost surely. To simplify the notation, introduce \( Y_n = (X_1, \ldots, X_n), \gamma_n = (\gamma_1, \ldots, \gamma_n) \) and

\[ M(\theta, \gamma_n) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(\gamma_i - \theta) \]

so that

\[ \tilde{\theta}_n = \Delta_{n+1} + \frac{1}{\sqrt{n}} M(\Delta_{n+1}, \gamma_n) \]  

for \( n \geq 1 \). Notice that \( \{Y_n\} \) is an \( m \)-dependent sequence. Let also \( \mathcal{F}_n \) be the \( \sigma \)-algebra generated by \( X_1, \ldots, X_n \). With

\[ \mathcal{C}_n(\theta, \gamma_n) = \text{Cor} \left[ M(\gamma_n, Y_n), M(\theta, Y_{n+1}) \right] \]

it may be shown that

\[ \sigma^2(\gamma) = \frac{1}{\gamma_0} \sum_{i=1}^{n} \mathcal{C}_i(\theta, \gamma) \]  

This relation will be useful here on in the proof. We will start by proving a series of lemmas. The proof of the last lemma is simple and therefore omitted. The proofs of Lemma 2 and 3 are similar to the proofs of Theorem 3.1 and Theorem 3.2 in Holst (1987).

**Lemma 1** Assume \( n \geq 2n \) and \( 0 < \gamma \leq n \). Then

\[ |\tilde{\theta}_n - \Delta_{n+1}| \leq Cn^{-1} \log n \]  

and

\[ \left| \frac{1}{\sqrt{n}} - \frac{1}{\gamma_{n+1}} \right| \leq Cn^{-1} \log n \]  

**Lemma 2**

\[ \Delta_n \rightarrow \theta \text{ as } n \rightarrow \infty. \]

**Proof.** Assume \( n \geq 2n \). After some manipulations, using (4.1) and \( E \left[ M(\theta, \gamma_n) | \mathcal{F}_n \right] = \rho = H_F(\theta, \gamma_n) \), we get
\[
\hat{\theta}_n - \theta = \hat{\theta}_{n-1} - \theta + \frac{1}{nb_{n-1}} M(\hat{\theta}_{n-1}, Y_n) \\
= \hat{\theta}_{n-1} - \theta + \frac{p - H_F(\hat{\theta}_{n-1})}{nb_{n-1}} + \frac{V_n}{nb_{n-1}} + w_n + R_n,
\]
with
\[
V_n = M(\hat{\theta}_{n-m}, Y_n) - E \left( M(\hat{\theta}_{n-m}, Y_n) | F_{n-m} \right),
\]
and
\[
w_n = \frac{1}{nb_{n-1}} \left( M(\hat{\theta}_{n-1}, Y_n) - M(\hat{\theta}_{n-m}, Y_n) \right).
\]

By Lemma 1 and (C), \(|R_n| \leq Cn^{-\zeta}\) for some \(\zeta > 3/2\), so
\[
\sum_{k=2n}^{\infty} |R_k| < \infty.
\]

Since \(V_n\) is adapted to \(F_n\) and \(E(V_n | F_{n-m}) = 0\), \(\{V_n\}_{2n}^{\infty}\) is a uniformly bounded sequence of mixingale differences. By McLeish (1975, Corollary (1.8)),
\[
\sum_{k=2n}^{\infty} \frac{V_k}{kb_{k-2n}^{*}} \text{converges},
\]
since \(\sum_{k} k^{-2} (\log k)^{2} < \infty\). By Lemma 1,
\[
\begin{cases}
|w_n| \leq Cn^{-1} \log n \\
P(w_n \neq 0) \leq Cn^{-\eta}(\log n)^{\eta} \\
P(w_n \neq 0, w_l \neq 0) \leq Cn^{-\eta}(\log n)^{\eta} l^{-\eta}(\log l)^{\eta} \text{for } |n - l| \geq m.
\end{cases}
\]

Actually, (4.12) implies that
\[
S_n = \sum_{k=2n}^{n} w_k \text{ converges.}
\]

Put \(S_n = S_n^+ - S_n^- = \sum_{k=2n}^{n} w_k^+ - \sum_{k=2n}^{n} w_k^-\), where \(w_k^+ = 0 \vee w_k\) and \(w_k^- = 0 \vee (-w_k)\). We will show that
\[
S_n^+ = \sum_{k=2n}^{n} w_k^+ \text{ converges.}
\]

The convergence of \(S_n^-\) is handled in the same way. First note that \(E(w_k^+) \leq Ck^{-1-\eta}(\log k)^{2}\) because of (4.12), so
\[
\sum_{k=2n}^{\infty} E(w_k^+) < \infty.
\]

After some calculation, it also follows from (4.12), if \(l > n\), that
\[
\text{Var}(S_l^+ - S_n^+) \leq Cn^{-2\eta + \varepsilon}
\]
for any \(\varepsilon > 0\). In connection with Chebyshev's inequality this gives
\[
\sum_{k=2n}^{\infty} P(|S_k^+ - S_n^+| \geq \varepsilon k^{-2}) < \infty
\]
for any $\varepsilon > 0$, so $S^n_k$ converges. Since $S^n_k$ is a non-decreasing sequence, (4.14) follows. Put now $h_k = h_k + V_n/(\text{rk}_n) + m$. Then, by (4.10), (4.11) and (4.12),

$$\sum_{k=0}^{\infty} h_k \text{ converges}.$$  

(4.17)

Choose now $a_n \to 0$ s.t.

$$\sum_{k=0}^{\infty} a_n/(\text{rk}_n) = \infty.$$  

(4.18)

This is possible since $h_n \leq e \log(n-1)$. Before $h = C_m$, with $C^n$ so large that $|a - \varepsilon| \geq h$, implies $|p - H(x)| \geq a$, for all but finitely many $n$. Then

$$\begin{align*}
K \rightarrow \theta \leq h_n \rightarrow k_n \rightarrow k_n \rightarrow k_n \rightarrow \theta + \frac{p_{n_0}}{n} + h_n, \\
K \rightarrow \theta \leq h_n \rightarrow k_n \rightarrow k_n \rightarrow \theta + \frac{p_{n_0}}{n} + h_n.
\end{align*}$$

Also, find $y_n \to 0$ s.t. $|x - \varepsilon| \leq h_n$, implies $|p - \theta (p - H(x)) (\text{rk}_n) + h_n| \leq y_n$. Then, for large enough $n$,

$$\begin{align*}
\{h_k - \theta\} &\leq |y_n h_n + \frac{p_{n_0}}{n} + h_n|, \\
\{h_k - \theta\} &\leq |y_n h_n + \frac{p_{n_0}}{n} + h_n|.
\end{align*}$$

The lemma now follows from (4.17), (4.18) and Lemma 1 in Derman and Sacks (1959).

Lemma 3

$h_n - b_q(f) \to 0$ as $n \to \infty$.

Proof. Let, for $x \in U$ (cf. (B)),

$$h(x) = n^{t} K(\varepsilon'(x - \varepsilon)) \, dR(x)$$

and

$$n_m = 2m - 1 \sum_{i=0}^{m-1} h_{i+1}(x_{i+1}).$$

Conditions (B), (D) and (E) imply $h_n(x) \to b_q(f)$ as $n \to \infty$ and $x \to y$. Hence, by Lemma 2,

$$h_n \to b_q(f)$$

(4.19)

Now

$$h_n - h_n = \frac{1}{n} \sum_{k=0}^{n-1} h_k + \frac{1}{n} \sum_{k=0}^{n-1} R_k,$$

with

$$U_n = \frac{1}{n} \sum_{k=0}^{n-1} \left(K' \left(K(b_q(X_k) - h_n) \right) - K' \left(K' \left(K(b_q(X_k) - h_n) \right) \right) \right)$$

and

$$R_n = \frac{1}{n} \sum_{k=0}^{n-1} \left(K' \left(K' \left(K(b_q(X_k) - h_n) \right) \right) \right) - K' \left(K(b_q(X_k) - h_n) \right).$$

Observe that $E(U_n) = 0$, so $(U_n)$ are martingale differences. McLeish (1975, Corollary 1.8) and Kruilik's Lemma gives

$$\frac{1}{n} \sum_{k=0}^{n-1} U_k = 0.$$
using the fact that $r < 1/2$. Finally, (E) and (4.4) imply $|k| < Ck^{1-r} \log 2$, which results in

$$\frac{1}{n} \sum_{k=1}^{n} |k| = 0.$$

### Lemma 4
For any $\delta < 1/2$,

$$a_i(\delta, \sigma) = 0.$$

**Proof.** In view of (B) and (4.6),

$$a_i(\delta, \sigma) = (n-1) \left( (k^0 - k^1) \frac{1}{\log k} + \frac{\delta}{\log n} + \frac{\delta}{\log 2} \right)$$

and $r_{\alpha, V_n}$ and $u_n$ as in Lemma 2. It follows as in Lemma 2 that the three sums $\sum_{k=1}^{n} k^0 r_{\alpha, V_n}$, $\sum_{k=1}^{n} k^{1-r} v_{\alpha, V_n}$, and $\sum_{k=1}^{n} v_{\alpha, u_n}$ converge as $n \to \infty$. (For the second sum, see Delic (1975), Corollary (1.8)) since $\sum_{n=1}^{\infty} k^{1-r} \log k < \infty$. The lemma now follows from Lemma 1 in Venter (1967) and the fact that

$$\lim_{n \to \infty} \frac{\ln |\gamma(\delta)|}{\ln n} = 0$$

by Lemma 3.

### Lemma 5
For some $c_2 > 0$,

$$a_i^*(\delta, \gamma(\delta)) = 0.$$

**Proof.** We will see below that the choice

$$c_2 \in \min \left( \frac{1}{n}, \frac{1}{2}, \frac{1}{2} \right)$$

will do, where $\delta$ is any admissible number in Lemma 4. Choose now $\delta_2 > 0$ so that supp($E$) $\subset [-\delta_2, \delta_2]$. Then, if $|x| \leq \delta_2$, it follows from Assumption (B) that

$$|k(x) - k(x)| = 0.$$

Hence, by (D) and Lemma 4,

$$a_i^*(\delta, \gamma(\delta)) = 0.$$

Next, by Kronecker's Lemma,

$$a_i^*(\delta, \gamma(\delta)) = 0,$$

provided $\sum_{n=1}^{\infty} (\ln n)/\left( n^{\alpha + \epsilon} \right)$ and $\sum_{n=1}^{\infty} (\ln n)/\left( n^{1-r} \right)$ converge. This follows as in the proof of Lemma 3, since $\sum_{n=1}^{\infty} (k^{1-r} \log k) < \infty$ and

$$\frac{1}{n} \sum_{k=1}^{n} |k| \leq Ck^{1-r} \log k - \infty.$$
Proof of Theorem 2: Define the sequence \(\{a_n\}_{n=1}^{\infty}\) through \(a_{n+1} - a_n = \frac{1}{a_n} - \frac{V_n}{N_{n-1}}\) and

\[
\delta_n - \delta = (a_{n+1} - a_n) \left(1 - \frac{1}{a_n} + \frac{V_n}{N_{n-1}}\right) \geq 0
\]

with \(k_n = \max(n, k^*)\). We will first show that \(\delta_n\) is asymptotically equivalent to \(\delta\), that is

\[
\delta_n \to 0,
\]

with \(\delta_n = \sqrt{a_n^*} - \delta\). Observe that \(\{\delta_n\}_{n=1}^{\infty}\) satisfy the recursion

\[
\delta_n = \left(1 - \frac{1}{a_n} \sqrt{a_n^*} - \sqrt{a_n^*} + \frac{1}{N_{n-1}} \frac{V_n}{N_{n-1}} + \frac{1}{a_n^*}(a_{n+1} - a)\right)
\]

with

\[
\delta_n = \frac{\delta(\delta_n) - \delta}{\delta_n^* - \delta}.
\]

As in the proof of Lemma 2, one shows that

\[
\sum_{n=1}^{\infty} \sqrt{a_n^*} \text{ converges}
\]

and

\[
\sum_{n=1}^{\infty} \sqrt{a_n} \text{ converges}
\]

By Lemma 3, \(\delta_n = \delta_n^* \) for all but finitely many \(\delta\). Hence,

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} - \frac{1}{a_n^*} \sqrt{\frac{V_n}{N_{n-1}}} \text{ converges}
\]

Lemma 2 and Condition (B) imply

\[
|a_n - 1| \leq C (|a_n^* - 1| + |\delta_n - \delta|^2).
\]

According to Lemma 4 this yields \(|1 - a_n| (\delta_n - \delta) \to 0\) for some \(\delta > 1/2\). Therefore,

\[
\sum_{n=1}^{\infty} \left|\frac{1 - a_n \delta}{\sqrt{\delta}}\right| < \infty.
\]

Now (4.21)-(4.24) and Lemma 1 of Venter (1967) imply (4.20). By Slutsky’s Lemma, it remains to prove asymptotic normality of \(\delta_n^*\). Observe that

\[
\delta_n - \delta = \frac{2n - 1}{N} (\frac{1}{a_{n+1} - a} - \delta) + \frac{1}{N} \sum_{n=1}^{\infty} \frac{V_n}{N_{n-1}}
\]

\[
= O(n^{-1}) + \frac{1}{N} \sum_{n=1}^{\infty} \frac{V_n}{N_{n-1}}
\]

Hence, it suffices to prove that

\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \frac{V_n}{N_{n-1}} \sim N(0, \sigma^2(y)).
\]
Actually, (4.25) follows from a Central Limit Theorem for mixingales in McLeish (1977), with \( X_n = i_1 Z_i \) if \( 2m \leq i \leq n \), \( X_n = 0 \) if \( 1 \leq i \leq 2m - 1 \), (4.1), \( \sum_{i=1}^n Z_i = \sum_{i=1}^n i_1 Z_i \), \( F_{X_n} = F_{i_1 Z_i} \), \( \alpha_4 = \alpha_4(i_1 Z_i) \) for \( 1 \leq i \leq m \) and \( \alpha_4 = \alpha_4 \geq \alpha_4(i_1 Z_i) \). Notice that \( \{X_n, \omega_n\} \) are uniformly bounded in \( n \) and \( \omega \), because of the choice of \( \{b_n\}_{n \in \mathbb{N}} \). Conditions (2.2)-(2.5) in McLeish (1977) are easily checked. It remains to check (2.6), which requires that for any \( s \), \( t \leq u < \infty \),

\[
E \left( \sum_{i=1}^{\lfloor s \rfloor} X_{i,n} \right)^2 - E \left( \sum_{i=1}^{\lfloor s \rfloor} X_{i,n} \right)^2 < \infty \text{ as } n \to \infty \tag{4.26}
\]

where \( \lfloor s \rfloor \) denotes the integer. Assume that \( n \) is so large that \( k_a(t) \geq 3m \) and \( k_a(t) - k_a(s) \geq 2m \).

In view of (4.3),

\[
E \left( \sum_{i=1}^{\lfloor s \rfloor} X_{i,n} \right)^2 - E \left( \sum_{i=1}^{\lfloor s \rfloor} X_{i,n} \right)^2 < \infty \text{ as } n \to \infty \tag{4.27}
\]

since \( \mathcal{C}(\delta, \theta) = 0 \) for \( |\theta| \geq \delta \) and \( \text{Cov}(X_{i,n}, X_{j,n}) = 0 \) for \( |i - j| > n \). To proceed further from (4.27), we will show below that

\[
\mathcal{C}(\delta, \theta) = \mathcal{C}(\delta, \theta)_{i, j} \to 0 \text{ as } n \to \infty \tag{4.28}
\]

for some \( \epsilon > 0 \) and with the \( \Omega \)-term holding uniformly for \( k_a(t) \leq u, \) \( l \leq j \leq k_a(t) \), \( |i - j| < m \). Now \( (x_1, x_2, x_3, x_4) = \mathcal{C}(\delta, \theta)_{i, j} \text{mod}(x_1, x_2, x_3, x_4) \) is a continuous and bounded function for \( n \in \mathbb{N} \). This is because \( \epsilon > 0 \) and since Condition (C) implies continuity of the \( \mathcal{C}_a \)-factor. Putting \( (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \), implies, via Lemma 2.3 and (4.24), that the BHS of (4.27) tends to zero as \( n \to \infty \). Finally, (4.28) is deduced by introducing

\[
X_n = \mathcal{C}(\delta, \theta)_{i, j} \text{mod}(x_1, x_2, x_3, x_4)
\]

if \( 2m \leq i \leq n \), with

\[
V_n = \mathcal{C}(\delta, \theta)_{i, j} \text{mod}(x_1, x_2, x_3, x_4)
\]

Then, because of the \( \mathcal{C}_a \)-dependence of the sequence \( \{Y_n\} \), and since \( k_a(t) \leq \min(i, j) \leq 2m \),

\[
\text{Cov}(X_{i,n} \text{mod}(x_1, x_2, x_3, x_4), X_{j,n} \text{mod}(x_1, x_2, x_3, x_4)) = \mathcal{C}(\delta, \theta)_{i, j} \text{mod}(x_1, x_2, x_3, x_4)
\]

when \( |i - j| < m \). Finally, the proof is completed by noting that

\[
\text{Cov}(X_{i,n} \text{mod}(x_1, x_2, x_3, x_4), X_{j,n} \text{mod}(x_1, x_2, x_3, x_4)) \leq \mathcal{C}(\delta, \theta)_{i, j} \text{mod}(x_1, x_2, x_3, x_4) \tag{4.29}
\]

for any \( r < \min(1 - \epsilon, \epsilon) \), which follows from Lemma 1 and Condition (C), using estimates similar to (4.12).

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References


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