Pfaffian quantum Hall state made simple: Multiple vacua and domain walls on a thin torus

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We analyze the Moore-Read Pfaffian state on a thin torus. The known sixfold degeneracy is realized by two inequivalent crystalline states with a four- and twofold degeneracy, respectively. The fundamental quasihole and quasiparticle excitations are domain walls between these vacua, and simple counting arguments give a Hilbert space of dimension 2^{n-1} for $2n-k$ holes and k particles at fixed positions and assign each a charge $\pm e/4$. This generalizes the known properties of the hole excitations in the Pfaffian state as deduced using conformal field theory techniques. Numerical calculations using a model Hamiltonian and a small number of particles support the presence of a stable phase with degenerate vacua and quarter-charged domain walls also away from the thin-torus limit. A spin-chain Hamiltonian encodes the degenerate vacua and the various domain walls.

One of the most intriguing aspects of the quantum Hall (QH) system is the possibility of experimentally observing non-Abelian statistics. In particular, it has been proposed that the fractional filling part of the observed[1] $\nu=5/2$ state is well described by the non-Abelian Moore-Read, or Pfaffian, wave function,[2]

$$\Psi_{\mathrm{Pf}}(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\Psi_{1/2}, \qquad (1)$$

where $\Psi_{1/2}$ is the bosonic Laughlin state at $\nu=1/2$ and z_i are the complex electron coordinates in the plane. By now, a great deal has been learned about (1) and its quasihole excitations, and we list some of the pertinent results. (1) Equation (1) is the exact ground state of a certain local three-body interaction.[3,4] (2) There are six degenerate ground states on a torus, and the electronic wave functions are explicitly known.[3] (3) The quasiholes have charge $e/4$ and can only be created in pairs.[1] The dimensionality of the Hilbert space for $2n$ holes at fixed positions in the plane is 2^{n-1}, and the wave functions in a particular "preferred basis" have been constructed.[5,6] (4) The quasihole wave functions are the conformal blocks of a correlator in a $c=3/2$ rational conformal field theory involving a bosonic vertex operator and a Majorana fermion. The conformal blocks have been explicitly constructed for four holes, where the Hilbert space is two dimensional. The braiding properties, or monodromies, of the conformal blocks translate into non-Abelian statistics for the quasiholes.[2,7] (5) The ground state in Eq. (1) can be viewed as a triplet pairing state of composite fermions, and the quasiholes as vortex excitations. The pairing picture nicely explains the presence of quarter-charged holes.[3,8]

Finally we should mention that the great recent interest in the non-Abelian QH states has to a large extent been spurred by the proposals to use them to build decoherence-free quantum computational devices.[9]

Recently, it was shown that studying the lowest Landau level (LLL) on a thin torus, with circumference L_1, allows for both a simple understanding of already established phenomena, and arriving at new results.[10,11] In particular, it was shown how the states in the Jain series[12] $\nu=p/(2pm+1)$ be-

come gapped crystals, with a unit cell of length $2pm+1$ (in units of the lattice spacing) and the fractionally charged excitations appear as domain walls between the $2pm+1$ different translational states of this crystal. In the infinitely thin limit, the $\nu=1/2$ state also forms a crystal, which, however, melts at $L_1 \sim 5.3$ (lengths are measured in units of the magnetic length), and a gapless homogeneous state of neutral fermions forms. All these properties are consistent with known properties of the bulk Laughlin and Jain states, and give a concrete realization of the dipole picture of the gapless $\nu=1/2$ state. There is strong evidence, analytical and numerical, that all these states develop continuously into the bulk states as $L_1 \rightarrow \infty$.[11,13,14] In summary, we consider it established that the main qualitative features of the bulk Laughlin and Jain states and the gapless $\nu=1/2$ state are present on the thin torus and that there is no phase transition as the two-dimensional bulk case is approached.

In light of the above, we have studied the torus generalization of the Pfaffian state (1) and its excitations on a thin torus. Despite recent progress in the construction of quasiparticle states,[15] much more is known about quasiholes than about quasiparticles in the QHE effect. However, our construction is manifestly particle-hole symmetric and allows for a unified description of quasiholes and quasiparticles. The analysis, to be given below, results in a simple and intuitive picture of the degenerate ground states, and the quasiholes and quasiparticles as domain walls between them. We obtain the general 2^{n-1}-fold degenerate state with k quasiholes and $2n-k$ quasiparticles. Using exact numerical diagonalization, we find that for a certain range of pseudopotential parameters these quarter-charged particles and holes are the lowest-energy excitations of systems with a small number of particles also at finite L_1.

Defining the magnetic translation operators $t_1=e^{(L_1/N_s)\partial_x}$, $t_2=e^{(L_2/N_s)(\partial_y+ix)}$, appropriate to the $A_y=0$ (Landau) gauge, a basis of lowest-Landau-level single-particle states on a torus (L_1,L_2) is given by $\psi_k=t_2^k\psi_0$, $k=0,1,\ldots,N_s-1$, where $\psi_0=\pi^{-1/4}L_1^{-1/2}\sum_n e^{inL_2x}e^{-(y+nL_2)^2/2}$. Here ψ_k is located along the line $y=-2\pi k/L_1$ and is a t_1 eigenstate, $t_1\psi_k=e^{i2\pi k/N_s}\psi_k$. The quantum number k thus labels both the position in the y direction and the momentum in the x direction. The many-

body translation operators $T_\alpha = \Pi_{i=1}^{N_e} t_{i\alpha}$ ($t_{i\alpha}$ translates electron i) commute with a translationally invariant electron interaction Hamiltonian H.

A general N_e-particle state in the LLL is a linear combination of states $\det(\psi_{k_1}(1)\cdots\psi_{k_{N_e}}(N_e))$, here labeled by $n_0 n_1 \cdots n_{N_s-1}$, where $n_k = 0,1$ and $\Sigma_{k=0}^{N_s-1} n_k = N_e$; T_2 generates translations: $T_2 n_0 n_1 \cdots n_{N_s-1} = n_{N_s-1} n_0 \cdots n_{N_s-2}$. As $L_1 \to 0$ hopping becomes unimportant and all energy eigenstates have the charges frozen in a regular lattice determined by the electrostatic interaction. In a half-filled Landau level the ground state is $101010\ldots$—this is the obvious one-dimensional limit when the electrons interact via a generic repulsive interaction. The interesting question now is what happens when the length L_1 moves away from zero. In Ref. [10] it was shown that, as hopping becomes more important, and for an unscreened Coulomb interaction at $\nu=1/2$, a gapless state obtained from the maximally hoppable state $01100110\ldots0110$ wins over the gapped crystal at $L_1 \approx 5.3$, and it was later shown[11] that the resulting Luttinger-liquid-type state is well described by a Fermi sea of composite fermions of the Rezayi-Read type.[16] From exact diagonalization studies using an unscreened Coulomb potential, one also learns that the gapless Rezayi-Read state is good at $\nu=1/2$, while at $\nu=5/2$ the gapped Pfaffian state is favored.[4,17] The difference between the two cases is due to the modifications in the short-distance interaction caused by the different one-particle states in the two Landau levels.

On the torus, the Pfaffian state is sixfold degenerate rather than only twofold as implied by the center-of-mass degeneracy. The technical reason for having the extra states is that on the torus, $\Psi_{\rm Pf} = {\rm Pf}\left(\frac{1}{z_i-z_j}\right)\Psi_{1/2} \to {\rm Pf}\left(\frac{\vartheta_a(z_i-z_j)}{\vartheta_1(z_i-z_j)}\right)\Psi_{1/2}^{(t)}$, where $\Psi_{1/2}^{(t)}$ is the torus version of $\Psi_{1/2}$, and $\vartheta_a(z)$ are Jacobi theta functions.[3] The extra threefold degeneracy corresponds to $a=2,3,4$. Since the Pfaffian state is gapped it is tempting to identify it with the crystalline state $A=010101\ldots01$ in the thin limit, but this, and its translated twin $T_2 A$, account for only two of the six ground states. Natural candidates for the other four are the four translations $T_2^k B$, $k=0,1,2,3$ of the state $B=01100110\ldots0110$. We have explicitly verified that these are the six ground states by projecting the Pfaffian state onto a single-particle basis and studying the thin limit: $a=2$ gives the two A states, whereas $a=3,4$ give the four B states. Note that all the six ground states have the property that any four adjacent sites are populated by exactly two particles, and that they are the unique states with this property.[21]

In a state formed by joining different ground states $ABAB\ldots$, domain walls with three and one electron on four adjacent sites, $AB \sim 1011$ and $BA \sim 0010$, respectively, are created,

$$AB = 01010\underline{1011}00110011001\underline{1}00101010101. \quad (2)$$

(Note that because of the periodic boundary conditions both domain walls are present in the AB state.) It follows from the Su-Schrieffer counting argument[18] that these domain walls have fractional charge $-e/4$ and $e/4$, respectively.[22] Thus the state $ABAB\ldots B$ has an alternating sequence of positive and negative quarter charges—AB, in particular, contains one quasiparticle-quasihole pair.

The fourfold degeneracy of B compared to the twofold degeneracy of A leads to a degeneracy of the states with four or more excitations. Imagine inserting B strings in a given A background. This can in general be done in two different ways as illustrated by the following example:

$$(ABAB)_1 = 0101\underline{01}100110\underline{01}01010101100110\underline{01}0101$$
$$\equiv 01010\underline{1011}00110\underline{01}01010101011001100101,$$

$$(ABAB)_2 = 01010\underline{1011}00110\underline{01}0101010101011001100101, \quad (3)$$

where the $\equiv$ sign denotes equality up to a total translation. The single particle-hole state AB in (2) is thus unique up to a translation, while the two states $(ABAB)_1$ and $(ABAB)_2$ cannot be translated into each other. Generalizing this we conclude that there are 2^{n-1} states of n particle-hole pairs at fixed positions. In comparing the two states above, we notice that they differ only in that one of the B segments is translated two lattice spacings. One might worry that this just corresponds to a shift of the positions of the domain walls, and would not imply the existence of many states at fixed positions. Note, however, that no combination of rigid translations and *local* motion of the domain walls (where they stay separate) can transform the states into each other, and thus they belong to different topological sectors.[23]

To obtain general states with quasiholes and/or quasiparticles one must insert extra empty sites and/or electrons. Define $A_0 = A0 = 0101\ldots010$ and $A_1 = 1A = 10101\ldots01$. The state A_0 has one extra 0 inserted—this excitation has, again by a straightforward counting argument, charge $e/2$. Similarly, A_1 has an excitation with charge $-e/2$. Joining these with B one obtains the new domain walls $A_0 B \sim 0100$, $BA_1 \sim 1101$ with charges $e/4$ and $-e/4$, respectively. The domain walls $BA_0 \sim BA \sim 0010$ and $A_1 B \sim AB \sim 1011$ are the same as those already present in AB [Eq. (2)]. Examples of a two-quasihole and of a two-quasiparticle state are

$$A_0 B = 0101\underline{0100}110011001100110\underline{0010}101,$$

$$A_1 B = 101010\underline{1011}0011001100\underline{110}101010. \quad (4)$$

A state with an arbitrary number of quarter-charged holes and particles, in arbitrary positions, can be formed as $X_1 B X_2 B \ldots X_n B$, where $X_i \in \{A, A_0, A_1\}$. Again disregarding a rigid translation, this state is 2^{n-1}-fold degenerate for fixed positions of the particles and holes.

In particular, the $2n$ quasihole states are $A_0 B A_0 B \ldots B$ with degeneracy 2^{n-1} as for the $2n$-hole Pfaffian state on the plane. For eight and 16 electrons, we have also explicitly verified that these states emerge as the leading terms in the thin-torus limit for the Pfaffian wave functions with two $e/4$ quasiholes [where $\vartheta_a(z_i-z_j)$ is replaced by $\vartheta_a(z_i-z_j+\frac{1}{2}(\eta_1-\eta_2))\vartheta_1(z_i-\eta_1)\vartheta_1(z_j-\eta_2)$ and the center-of-mass coordinate becomes $Z=\Sigma_i z_i + \frac{1}{4}(\eta_1+\eta_2)$ (Ref. [3])].

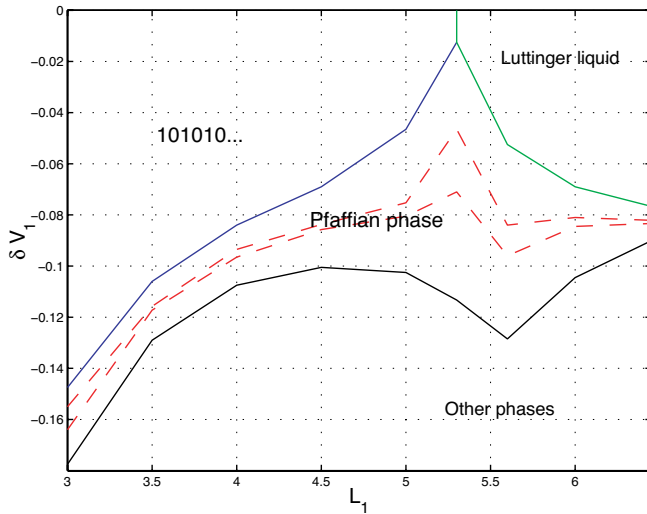The six Pfaffian states are the exact ground states of a hyperlocal three-body interaction on the torus[4]—this holds

FIG. 1. (Color online) Phase diagram for a half-filled Landau level on the torus as a function of L_1 and the pseudopotential parameter δV_1 ($\delta V_1 = 0$ corresponds to Coulomb interaction in the lowest LL). Results are obtained using exact diagonalization for eight electrons. The Pfaffian phase is found in the central part (solid lines) of the diagram. This is the region where the six lowest-lying states are those that continuously approach the crystals A and B. Within the dashed lines, these states are almost degenerate—they differ in energy by less than 10% of the gap to the next state. In the Pfaffian phase we observe quasiparticles and quasiholes of charge $\pm e/4$. In the top left the ground state is $101010\cdots$ and the phase in the top right is the gapless Luttinger liquid phase described in Ref. 10.

for general L_1 as it depends on the local properties only. The lattice Hamiltonian[24] takes the form $H_3 = \Sigma_{\{k_i\}} V_{\{k_i\}} c^\dagger_{k_1} c^\dagger_{k_2} c^\dagger_{k_3} c_{k_4} c_{k_5} c_{k_6}$ with

$$V_{\{k_i\}} \propto \delta_{k_{123}, k_{456}} k_{12} k_{13} k_{23} k_{45} k_{46} k_{56}$$
$$\times \exp\left\{ -2\pi^2 \left[\sum_i k_i^2 - \frac{1}{6} \left(\sum_i k_i \right)^2 \right] \middle/ L_1^2 \right\},$$

where $k_{ij} = k_i - k_j$ and $k_{ijk} = k_i + k_j + k_k$. In the thin-torus limit, this implies that the electrostatic energy is minimized by minimizing the number of sequences of four consecutive sites containing three electrons (or holes). The six states A and B above are the unique states at half filling that have no such sequences.[25] Such sequences of electrons (holes) are also absent from the states with quasiholes (quasiparticles).

We have performed exact diagonalization studies of small systems that corroborate the picture given above. Following Rezayi and Haldane[4] we consider the electron gas on the torus as a function of the pseudopotential parameter δV_1 and find that the six Pfaffian states are favored for a finite range in parameter space; see Fig. 1. In particular, as the torus becomes thin, these states continuously approach the crystal-line states proposed above. Exactly at half filling we find that the low-lying excited energy states are consistent with the creation of a single quasiparticle-quasihole pair. Due to their opposite charge they attract each other and the very lowest energies are obtained when they overlap. However, the entire

low-energy spectrum is built up of states with different separations between the particle-hole pair. Slightly away from half filling we find ground states that have well-separated quasihole or quasiparticle excitations upon a Pfaffian background. In systems with two quasiholes, the formation of $e/4$ charges is very clear and also stable as L_1 increases from zero. As an example we find that the ground state of the $\nu = 8/17$ system with just eight particles evolves continuously from $01100110\underline{0010}101010$ (i.e., a state with two $e/4$ quasiholes as far apart as possible) into a charge-density-wave state with the same symmetry. This density wave is seen clearly throughout the Pfaffian phase in Fig. 1. Of course, we find—consistent with particle-hole symmetry—the same scenario in terms of $-e/4$ quasiparticles for $\nu = 9/17$. Furthermore, we find that the lowest-lying excitations of these systems are those that move the $\pm e/4$ charges closer together.

In Ref. 10 it was shown that for $\nu = 1/2$, and a particular choice of short-range Hamiltonian relevant for a thin torus the system can be written as a spin-half XY chain by the mapping $10 \rightarrow \uparrow$, $01 \rightarrow \downarrow$. A spin flip $\uparrow\downarrow \leftrightarrow \downarrow\uparrow$ corresponds to the nearest-neighbor hopping $1001 \leftrightarrow 0110$ and the ground state emerges from the maximally hoppable state B. By standard techniques the spin chain can be mapped onto free fermions, and it is natural to assume that a more general Hamiltonian will correspond to a Luttinger liquid.

The most obvious way to generalize this description to the Pfaffian state is to consider the phase diagram for the spin-half model in the presence of anisotropic and more long-range interactions. In addition to the gapless Luttinger liquid phase there are at least two gapped phases: the Ising phase, which will always win in the extreme thin-torus limit, and a spin-Peierls phase. The latter is, however, not a candidate for the Pfaffian state; the spin pairing breaks translational invariance, the ground-state degeneracy on a torus is not six, and there are no quarter-charged holes. It is an open question whether there is a QH counterpart to the spin-Peierls state.

The origin of these difficulties is that the above mapping of two sites to a single spin does not allow for domain walls. To overcome this, we map each site to a spin such that the occupation number gives the z component, $\sigma^z_i = 2n_i - 1$. Remembering that the ground states are the states where any four adjacent sites have exactly two particles, suggests the Hamiltonian $H_p = V \Sigma_i (\sigma^z_i + \sigma^z_{i+1} + \sigma^z_{i+2} + \sigma^z_{i+3})^2$, where $V > 0$, which clearly has the correct ground states. A quarter-charged quasiparticle (hole) has one quadruple of sites with three electrons (holes); hence its excitation energy is $4V$. This spin model is a frustrated antiferromagnetic spin chain, as can be seen by rewriting the Hamiltonian as $H_p = 2V \Sigma_i (3\sigma^z_i \sigma^z_{i+1} + 2\sigma^z_i \sigma^z_{i+2} + \sigma^z_i \sigma^z_{i+3}) + \text{const}$. The kinetic Hamiltonian $0110 \leftrightarrow 1001$ is unfortunately somewhat complicated, $H_k = t \Sigma_i (\sigma^+_i \sigma^-_{i+1} \sigma^-_{i+2} \sigma^+_{i+3} + \text{H.c.})$. The hopping term lifts the degeneracy of the six ground states—this can, however, be compensated for by fine-tuning the $\sigma^z_i \sigma^z_j$ couplings.

In summary, we have presented a simple way to understand the vacuum degeneracy and the $\pm e/4$-charged quasiparticles and holes of the Pfaffian wave function in the thin-torus limit. We have also given numerical evidence for this fractionalized phase to survive as the torus becomes thicker. In particular we found that the internal Hilbert space of a

configuration of $2n$ particles and/or holes is 2^{n-1}, and we should again stress that the quasiparticles enter in a natural way in our description. That the degeneracy of the internal quasihole Hilbert space agrees with the bulk state strongly suggests that the non-Abelian statistics also is present in the thin-torus limit. Since the configuration space is one dimensional and discrete, it is not clear how to define non-Abelian statistics, but we might speculate that it would be encoded in properties of the (rather complicated) spin chain defined above.

Recently we became aware that Haldane has obtained results similar to those presented here,[19] and a closely related study of the bosonic Pfaffian state at $\nu=1$ appeared shortly after our work.[20]

[1] R. Willet, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).

[2] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).

[3] M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991); Nucl. Phys. B **374**, 567 (1992).

[4] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).

[5] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).

[6] N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996).

[7] B. Blok and X. G. Wen, Nucl. Phys. B **374**, 615 (1992).

[8] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).

[9] See, e.g., M. H. Freedman, A. Kitaev, M. Larsen, and Z. Wang, Bull., New Ser., Am. Math. Soc. **40**, 31 (2003); A. Kitaev, Ann. Phys. **303**, 2 (2003).

[10] E. J. Bergholtz and A. Karlhede, Phys. Rev. Lett. **94**, 026802 (2005).

[11] E. J. Bergholtz and A. Karlhede, J. Stat. Mech.: Theory Exp. 2006, L04001.

[12] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

[13] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. B **50**, 17199 (1994).

[14] A. Seidel, H. Fu, D.-H. Lee, J. M. Leinaas, and J. Moore, Phys. Rev. Lett. **95**, 266405 (2005).

[15] T. H. Hansson, C. C. Chang, J. K. Jain, and S. Viefers, cond-mat/0603125 (unpublished).

[16] E. Rezayi and N. Read, Phys. Rev. Lett. **72**, 900 (1994).

[17] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).

[18] W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. **46**, 738 (1981).

[19] F. D. M. Haldane (unpublished).

[20] A. Seidel and D.-H. Lee, cond-mat/0604465 (unpublished).

[21] The Pfaffian states on the torus and the crystal states A and B have the same quantum numbers. The simultaneously conserved symmetries are generated by T_1 and T_2^2 and their eigenstates are A and $B_\pm = B \pm T_2^2 B$. It is easy to see that $B_\pm$ can be used in the construction of the domain walls instead of B and $T_2^2 B$.

[22] Alternatively, notice that moving the primed electron in the configuration 0101<u>0110</u>01100110 a single site gives 01010101<u>0110</u>0110, i.e., the domain wall moves four sites and thus carries charge $-e/4$.

[23] Having an exact definition of the position of a domain wall is not necessary for these arguments, but there is a rather natural prescription using the $B_\pm$ states introduced in Ref. [19]. With this definition the state counting for fixed domain wall positions agrees with the one in the text, which is, however, easier to visualize.

[24] We give the matrix elements for the cylinder for simplicity.

[25] A actually has lower electrostatic energy than B—even as $L_1 \to 0$ there is a contribution from the hopping terms that makes the total energies for A and B equal. This is an artifact of the hyperlocal interaction; for a more realistic longer-range interaction, hopping freezes out completely, leaving an entirely electrostatic problem.