TOPICS IN DISCRETE PROBABILITY THEORY

DANIEL AHLBERG

Discrete probability theory may refer to the classical branch of probability theory concerned with experiments with a finite or countable number of outcomes. However, discrete probability theory also refers to the vivid area of research in contemporary probability theory that encompasses a wide range of stochastic processes with a discrete component. This includes card shufflings, random graphs, random walks and other random processes with a spatial component. Below we give a brief description of some of these topics that may serve as inspiration for master students looking for project topic. Students interested in exploring either of these topics (or other related topic) further are welcome to get in touch to discuss supervision.

1. Percolation theory

Percolation theory has its origins in the 1950s. It started out with a model for the flow of a fluid through a porous material, and quickly evolved into a proper area of research. As a motivating example, one could be interested in the design of a membrane able to block large molecules, while allowing smaller ones to pass through. In order to do so, one is interested in understanding how the local structure of the material influences its global properties.

A basic mathematical model can be described as follows: Consider a chessboard of size $n \times n$, thus consisting of n^2 squares in total. For each square of the board, toss a (possibly biased) coin to determine whether the square should be coloured black or white. We may think of black squares as 'open', through which a fluid may pass through, and white squares as 'closed'. Let A denote the event that it is possible to cross the chessboard from its left side to its right, without stepping on white squares. (For instance, we may allow moves to nearest neighbours, but disallow diagonal steps, corresponding to the eligible moves of a rook.) The local structure of the resulting colouring is governed by the probability p that a square is coloured black, and one may be interested in how the probability of the event A is affected as the parameter p is varied. Clearly, the probability of A must be increasing in terms of p. It turns out that there exists a *critical* value $p_c \in (0, 1)$ such that for $p < p_c$ the probability of the event A turns rapidly to zero as $n \to \infty$, and for $p > p_c$ the probability of A tends rapidly to one as $n \to \infty$. Describing the behaviour of the model at the critical value $p = p_c$ is the most challenging and interesting.

2. Analysis of Boolean functions

Consider an election between two candidates. The majority rule, i.e. the voting rule where the candidate that gains the majority of the votes is elected, is generally considered the most adequate. But, on what basis is that? Is it possible to justify this use of the majority rule mathematically? It is indeed, and the justification is based on the analysis of Boolean functions, i.e. functions of the form $f : \{0, 1\}^n \to \{0, 1\}$.

There are many fascinating phenomena related to Boolean functions. One aspect that has received considerable attention in the past couple of decades is that of *sensitivity* and *stability* of Boolean function with respect to small perturbations in the input variables. The majority rule is stable in this regard, which implies that the outcome of an election is thus unlikely to be affected by a small proportion of votes being recorded incorrectly. In fact, one can show that the majority rule is the most stable voting rule among a large class of voting rules. Many other interesting Boolean functions, in particular Boolean functions related to percolation crossings as describes above, turn out to be sensitive with respect to small perturbations.

3. Spatial growth and competition

The $\mathbb{Z}^2$ *lattice* is the infinite graph whose vertex set consists of all points in $\mathbb{R}^2$ with integer coordinates, and where any two vertices at distance 1 are connected by an edge. Assign exponentially distributed random variables (with mean 1) to the edges of the graph. The random variables can be thought of as lengths of the edges, and the resulting weighted graph thus induces a (random) metric on $\mathbb{Z}^2$ as follows: The distance $T(y, z)$ between two points y and z is given by the minimal sum of the variables along any path connecting the two points, i.e.

$$T(y, z) := \inf \Big\{ \sum_{e \in \pi} \omega_e : \pi \text{ is a path connecting } y \text{ to } z \Big\},$$

where ω_e is the variable associated to the edge e. Of particular interest is to describe the large-scale behaviour of distances, balls and geodesics in the resulting random metric space.

The above construction is usually referred to as *first-passage percolation*, and is commonly interpreted as a model for spatial growth: Let, initially, a growing entity occupy the origin $(0, 0) \in \mathbb{Z}^2$. As time evolves, occupied sites occupy their vacant neighbours at rate 1. (That is, occupation times are exponentially distributed time with mean 1.) Once a site is occupied it remains occupied forever. The set of vertices that are occupied at time t corresponds precisely to the ball of radius t centered at the origin in the random metric described above.

The model for spatial growth may be extended to a model for competing growth as follows: Initially, colour the origin $(0, 0)$ red and let its neighbour $(1, 0)$ blue, and let the two colours spread to uncoloured sites according to the before mentioned rule. (Once a site has been coloured it will keep that colour forever.) The two types are said to *coexist* if the eventual colouring of $\mathbb{Z}^2$ consists of infinitely many sites of each colour. A famous theorem states that coexistence will occur with positive probability, and an even more famous conjecture states that if one of the two types were to grow at a faster rate $\lambda > 1$, then coexistence would not be possible. The conjecture remains unsettled, but it is known to be true if the growth of the two types is restricted to the upper half-plane, as opposed to the whole plane.